Mathematics

(Chapter – 6) (Application of Derivatives) (Exercise 6.1)
(Class – XII)

Question 1:

Find the rate of change of the area of a circle with respect to its radius r when

(a) $r = 3cm$ **(b)** $r = 4\ cm$

Answer 1:

Area of circle with radius r is given by $A = \pi r^2$

Therefore, the rate of change of A with respect to $r = \frac{dA}{dr} = 2\pi r$

(a) When $r = 3\ cm$, we have $\frac{dA}{dr} = 6\pi$

Hence, the area of circle is changing at the rate of $6\pi\ cm^2/cm$.

(b) When $r = 4\ cm$, we have $\frac{dA}{dr} = 8\pi$

Hence, the area of circle is changing at the rate of $8\pi\ cm^2/cm$.

Question 2:

The volume of a cube is increasing at the rate of 8 cm3/s. How fast is the surface area increasing when the length of an edge is 12 cm?

Answer 2:

Let the length of edge of cube $= x\ cm$, therefore, the volume of cube $V = x^3$

So, the rate of change of V with respect to $t = \frac{dV}{dt} = 3x^2\frac{dx}{dt}$

Given that: $\frac{dV}{dt} = 8\ cm^3/s$, therefore, $8 = 3x^2\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = \frac{8}{3x^2}$ ... (1)

Surface area of cube $S = 6x^2$

The rate of change of S with respect to $t = \frac{dS}{dt} = 12x\frac{dx}{dt}$

Putting the value of $\frac{dx}{dt}$ from equation (1), we have

$$\frac{dS}{dt} = 12x\frac{8}{3x^2} = \frac{32}{x}$$

When $x = 12\ cm$, we have $\frac{dS}{dt} = \frac{32}{12} = \frac{8}{3}$

Hence, the surface area increasing at the rate of $\frac{8}{3}\ cm^2/s$.

Question 3:

The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

Answer 3:

Let the radius of circle $= r\ cm$, so area of circle with radius r is given by $A = \pi r^2$

Therefore, the rate of change of A with respect to $t = \frac{dA}{dt} = 2\pi r.\frac{dr}{dt} = 2\pi r.3 = 6\pi r$

When $r = 10\ cm$, then $\frac{dA}{dt} = 6\pi.(10) = 60\pi\ cm^2/s$

Hence, the area of circle is increasing at the rate of $60\pi\ cm^2/s$.

Question 4:

An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

Answer 4:

Let the side of the cube $= x\ cm$, therefore, the volume of cube $V = x^3$

The rate of change of V with respect to $t = \frac{dV}{dt} = 3x^2\frac{dx}{dt}$

Given that: the side of the cube is increasing at the rate of $3\ cm/s$, therefore, $\frac{dV}{dt} = 3x^2(3) = 9x^2$

When $x = 10\ cm$, then $\frac{dV}{dt} = 9(10)^2 = 900\ cm^3/s$

Hence, the volume of the cube is increasing at the rate of $900\ cm^3/s$.

Question 5:
A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

Answer 5:

Let the radius of circle $= r\ cm$

Area of circle with radius r is given by $A = \pi r^2$

Therefore, the rate of change of A with respect to $t = \frac{dA}{dt} = 2\pi r.\frac{dr}{dt} = 2\pi r.5 = 10\pi r$

When $r = 8\ cm$, then $\frac{dA}{dt} = 10\pi.(8) = 80\pi\ cm^2/s$

Hence, the enclosed area is increasing at the rate of $80\pi\ cm^2/s$.

Question 6:
The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

Answer 6:

Let the radius of circle $= r\ cm$, Circumference of circle $C = 2\pi r$

Therefore, the rate of change of C with respect to $t = \frac{dC}{dt} = 2\pi.\frac{dr}{dt} = 2\pi.(0.7) = 1.4\pi$

When $r = 4.9\ cm$, then $\frac{dC}{dt} = 1.4\pi = 1.4 \times \frac{22}{7} = 4.4\ cm/s$

Hence, the circumference is increasing at the rate of $4.4\ cm/s$.

Question 7:
The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When x = 8 cm and y = 6 cm, find the rates of change of **(a)** the perimeter, and **(b)** the area of the rectangle.

Answer 7:

Length of rectangle $= x\ cm$ and breadth $= y\ cm$

Given: $\frac{dx}{dt} = -5$ cm/min and $\frac{dy}{dt} = 4\ cm/min$

(a) Perimeter of rectangle $P = 2(x + y)$

Therefore, the rate of change of P with respect to $t = \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-5 + 4) = -2\ cm/min$

When $x = 8\ cm$ and $y = 6\ cm$, then $\frac{dP}{dt} = -2\ cm/min$

Hence, the perimeter is decreasing at the rate of $2\ cm/min$.

(b) Area of rectangle $A = xy$

Therefore, the rate of change of A with respect to $t = \frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt} = -5y + 4x\ cm^2/min$

When $x = 8\ cm$ and $y = 6\ cm$, then $\frac{dA}{dt} = -5 \times 6 + 4 \times 8 = 2\ cm^2/min$

The area of rectangle is increasing at the rate of $2\ cm^2/min$.

Question 8:
A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

Answer 8:

Let the radius of spherical balloon $= r\ cm$ therefore the volume of balloon $V = \frac{4}{3}\pi r^3$

Therefore, the rate of change of V with respect to $t = \frac{dV}{dt} = \frac{4}{3}\pi.3r^2\frac{dr}{dt} \Rightarrow 900 = 4\pi r^2\frac{dr}{dt}$

When $r = 15\ cm$, then $900 = 4\pi(15)^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{900}{4\pi \times 225} = \frac{1}{\pi}\ cm/s$

Hence, the radius of balloon is increasing at the rate of $\frac{1}{\pi}\ cm/s$.

Question 9:

A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the radius is 10 cm.

Answer 9:

Let the radius of balloon $= r\ cm$, therefore the volume of balloon $V = \frac{4}{3}\pi r^3$

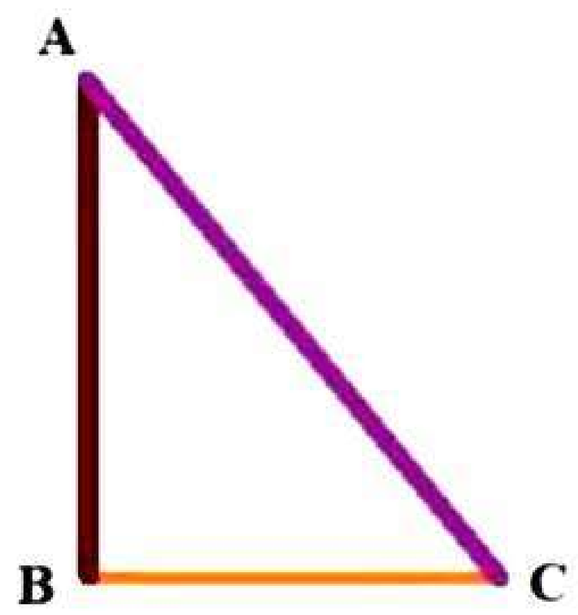
Therefore, the rate of change of V with respect to $r = \frac{dV}{dr} = \frac{4}{3}\pi . 3r^2 = 4\pi r^2$

When $r = 10\ cm$, then $\frac{dV}{dr} = 4\pi(10)^2 = 400\pi\ cm^3/cm$

Hence, the volume of balloon is increasing at the rate of $400\pi\ cm^3/cm$.

Question 10:

A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?



Answer 10:

In figure, AB is wall and AC is ladder. Let, BC $= x$ and AB $= y$.

Given: $\frac{dx}{dt} = 2\ cm/s$ and $AC = 5\ cm$

In triangle ABC, $AC^2 = AB^2 + BC^2 \Rightarrow 25 = y^2 + x^2$

When $x = 4\ cm$, then $25 = y^2 + 16 \quad \Rightarrow y^2 = 9 \quad \Rightarrow y = 3$

Differentiating with respect to t, we have $0 = 2y\frac{dy}{dt} + 2x\frac{dx}{dt}$

When $x = 4\ cm$ and $y = 3$, then, $0 = 2(3)\frac{dy}{dt} + 2(4)(2) \Rightarrow \frac{dy}{dt} = -\frac{16}{6} = -\frac{8}{3}\ cm/s$

Hence, the height is decreasing at the rate of $\frac{8}{3}\ cm/s$.

Question 11:

A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

Answer 11:

Given that: $\frac{dy}{dt} = 8\frac{dx}{dt}$ ... (1)

and $6y = x^3 + 2$ ... (2)

Differentiating with respect to t, we have

$6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$ ... (3)

Solving (1) and (3), we have

$$6\left(8\frac{dx}{dt}\right) = 3x^2\frac{dx}{dt}$$

$\Rightarrow 3x^2 = 48$

$\Rightarrow x^2 = 16 \quad \Rightarrow x = \pm 4$

If $x = 4$, then form equation (2), we have

$6y = 4^3 + 2 \quad \Rightarrow 6y = 66 \quad \Rightarrow y = 11$,

Therefore, the point $= (4, 11)$

If $x = -4$, then from equation (2), we have

$6y = -4^3 + 2 \quad \Rightarrow 6y = -62 \quad \Rightarrow y = -\frac{31}{3}$, therefore, the point $= \left(-4, -\frac{31}{3}\right)$

Hence, the required points are $(4,11)$ and $\left(-4, -\frac{31}{3}\right)$.

Question 12:

The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?

Answer 12:

Let the radius of bubble $= r\ cm$, therefore the volume $V = \frac{4}{3}\pi r^3$

Therefore, the rate of change of V with respect to $t = \frac{dV}{dt} = \frac{4}{3}\pi.3r^2\frac{dr}{dt} \quad = 4\pi r^2\left(\frac{1}{2}\right) = 2\pi r^2$

When $r = 1\ cm$, then $\frac{dV}{dt} = 2\pi(1)^2 = 2\pi\ cm^3/s$

Hence, the volume of the bubble is increasing at the rate of $2\pi\ cm^3/s$.

Question 13:

A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.

Answer 13:

Radius of balloon $= \frac{3}{4}(2x+1)\ cm$, therefore volume $V = \frac{4}{3}\pi\left[\frac{3}{4}(2x+1)\right]^3$

Therefore, the rate of change of V with respect to $x = \frac{dV}{dx} = \frac{4}{3}\pi.3\left[\frac{3}{4}(2x+1)\right]^2\frac{3}{4}(2) = \frac{27}{8}\pi(2x+1)^2$

Hence, the rate of change of volume of balloon is $\frac{27}{8}\pi(2x+1)^2$.

Question 14:

Sand is pouring from a pipe at the rate of 12 cm3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

Answer 14:

Let the radius of cone $= r\ cm$ and height $= h\ cm$

Given that: The rate of change of Volume $V = \frac{dV}{dt} = 12\ cm^3/s$ and $h = \frac{1}{6}r$

Volume of cone $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3$

Therefore, the rate of change of V with respect to $t = \frac{dV}{dt} = 12\pi.3h^2\frac{dh}{dt}$

$$\Rightarrow 12 = 36\pi h^2\frac{dh}{dt} \qquad \Rightarrow \frac{dh}{dt} = \frac{1}{3\pi h^2}$$

When $h = 4\ cm$, then $\frac{dh}{dt} = \frac{1}{3\pi(4)^2} = \frac{1}{48\pi}\ cm/s$

Hence, the height of the sand cone is increasing at the rate of $\frac{1}{48\pi}\ cm/s$.

Question 15:

The total cost C(x) in Rupees associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000.$$

Find the marginal cost when 17 units are produced.

Answer 15:

Total cost $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$

Therefore, the marginal cost $\frac{dC}{dx} = 0.021x^2 - 0.006x + 15$

When $x = 17$, then $\frac{dC}{dx} = 0.021(17)^2 - 0.006(17) + 15 = 6.069 - 0.102 + 15 = 20.967$

Hence, the marginal cost is ₹20.967, when 17 units are produced.

Question 16:

The total revenue in Rupees received from the sale of x units of a product is given by

$$R(x) = 13x^2 + 26x + 15.$$

Find the marginal revenue when $x = 7$.

Answer 16:

Total revenue $R(x) = 13x^2 + 26x + 15$

Therefore, the marginal revenue $\frac{dR}{dx} = 26x + 26$

When $x = 7$, then $\frac{dR}{dx} = 26(7) + 26 = 182 + 26 = 208$

Hence, the marginal revenue is ₹208, when x = 7.

Choose the correct answer in the Exercises 17 and 18.

Question 17:

The rate of change of the area of a circle with respect to its radius r at $r = 6$ cm is:

(A) 10π (B) 12π (C) 8π (D) 11π

Answer 17:

Area of circle $A = \pi r^2$,

The rate of change of A with respect to $r = \frac{dA}{dr} = \pi.2r \quad = 2\pi r$

When $r = 6\ cm$, then $\frac{dA}{dt} = 2\pi(6) = 12\pi \frac{cm^2}{cm}$.

Hence, the option (B) is correct.

Question 18:

The total revenue in Rupees received from the sale of x units of a product is given by

$$R(x) = 3x^2 + 36x + 5.$$

The marginal revenue, when $x = 15$ is

(A) 116 (B) 96 (C) 90 (D) 126

Answer 18:

Total revenue $R(x) = 3x^2 + 36x + 5$,

Therefore, the marginal revenue $\frac{dR}{dx} = 6x + 36$

When $x = 15$, then $\frac{dR}{dx} = 6(15) + 36 = 90 + 36 = 126$.

Hence, the option (D) is correct.

Mathematics

(Chapter – 6) (Application of Derivatives) (Exercise 6.2)
(Class – XII)

Question 1:

Show that the function given by $f(x) = 3x + 17$ is strictly increasing on **R**.

Answer 1:

$f(x) = 3x + 17 \quad \Rightarrow f'(x) = 3 > 0$, For all $x \in \mathbf{R}$

Therefore, the function is strictly increasing on **R**.

Question 2:

Show that the function given by $f(x) = e^{2x}$ is strictly increasing on **R**.

Answer 2:

$f(x) = e^{2x} \quad \Rightarrow f'(x) = 2e^{2x} > 0$, For all $x \in \mathbf{R}$

Therefore, the function is strictly increasing on **R**.

Question 3:

Show that the function given by $f(x) = \sin x$ is

(a) strictly increasing in $\left(0, \frac{\pi}{2}\right)$ **(b)** strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(c) neither increasing nor decreasing in $(0, \pi)$.

Answer 3:

$f(x) = \sin x \quad \Rightarrow f'(x) = \cos x$

(a) $f'(x) = \cos x > 0$, For all $x \in \left(0, \frac{\pi}{2}\right)$

Therefore, the function is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

(b) $f'(x) = \cos x < 0$, For all $x \in \left(\frac{\pi}{2}, \pi\right)$

Therefore, the function is strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

(c) $f'(x) = \cos x > 0$, For all $x \in \left(0, \frac{\pi}{2}\right)$ and $f'(x) = \cos x < 0$, For all $x \in \left(\frac{\pi}{2}, \pi\right)$

Therefore, the function is neither increasing nor decreasing on $(0, \pi)$.

Question 4:

Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is

(a) strictly increasing **(b)** strictly decreasing

Answer 4:

$f(x) = 2x^2 - 3x \quad \Rightarrow f'(x) = 4x - 3$

If $f'(x) = 0 \quad \Rightarrow 4x - 3 = 0 \quad \Rightarrow x = \frac{3}{4}$

(a) $f'(x) = 4x - 3 > 0$, For all $x \in \left(\frac{3}{4}, \infty\right)$, therefore, the function is strictly increasing on $\left(\frac{3}{4}, \infty\right)$.

(b) $f'(x) = \cos x < 0$, For all $x \in \left(-\infty, \frac{3}{4}\right)$, therefore, the function is strictly decreasing on $\left(-\infty, \frac{3}{4}\right)$.

Question 5:

Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is

(a) strictly increasing **(b)** strictly decreasing

Answer 5:

$f(x) = 2x^3 - 3x^2 - 36x + 7 \Rightarrow f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)) = 6(x - 3)(x + 2)$

If $f'(x) = 0 \quad \Rightarrow 6(x - 3)(x + 2) = 0 \quad \Rightarrow x = -2, 3$

$x = -2$ and $x = 3$ divides the real number line into three intervals $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.

Intervals	Sign of $f'(x)$	Nature of the function f
$(-\infty, -2)$	$(-)(-) > 0$	f is increasing
$(-2, 3)$	$(-)(+) < 0$	f is decreasing
$(3, \infty)$	$(+)(+) > 0$	f is increasing

(a) Function f is increasing on $(-\infty, -2)$ and $(3, \infty)$.

(b) Function f is decreasing on $(-2, 3)$.

Question 6:

Find the intervals in which the following functions are strictly increasing or decreasing:

(a) $f(x) = x^2 + 2x + 5$

(b) $f(x) = 10 - 6x - 2x^2$

(c) $f(x) = -2x^3 - 9x^2 - 12x + 1$

(d) $f(x) = 6 - 9x - x^2$

(e) $f(x) = (x+1)^3(x-3)^3$

Answer 6:

(a) $f(x) = x^2 + 2x + 5$ $\quad \Rightarrow f'(x) = 2x + 2$

If $f'(x) = 0 \quad \Rightarrow 2x + 2 = 0 \quad \Rightarrow x = -1$

$f'(x) = 2x + 2 < 0$, For all $x \in (-\infty, -1)$, therefore, the function is strictly decreasing on $(-\infty, -1)$.

$f'(x) = 2x + 2 > 0$, For all $x \in (-1, \infty)$, therefore, the function is strictly increasing on $(-1, \infty)$.

(b) $f(x) = 10 - 6x - 2x^2$ $\quad \Rightarrow f'(x) = -6 - 4x$

If $f'(x) = 0 \quad \Rightarrow -6 - 4x = 0 \quad \Rightarrow x = -\frac{3}{2}$

$f'(x) = -6 - 4x > 0$, For all $x \in \left(-\infty, -\frac{3}{2}\right)$, therefore, the function is strictly increasing on $\left(-\infty, -\frac{3}{2}\right)$.

$f'(x) = -6 - 4x < 0$, For all $x \in \left(-\frac{3}{2}, \infty\right)$, therefore, the function is strictly decreasing on $\left(-\frac{3}{2}, \infty\right)$.

(c) $f(x) = -2x^3 - 9x^2 - 12x + 1$

$\Rightarrow f'(x) = -6x^2 - 18x - 12 = -6(x^2 + 3x + 2)) = -6(x+2)(x+1)$

If $f'(x) = 0 \quad \Rightarrow -6(x+2)(x+1) = 0 \quad \Rightarrow x = -2, -1$

$x = -2$ and $x = -1$ divides the number line into three intervals $(-\infty, -2)$, $(-2, -1)$ and $(-1, \infty)$.

Interval	Sign of $f'(x)$	Nature of f
$(-\infty, -2)$	$(-)(-) > 0$	f is increasing
$(-2, -1)$	$(+)(-) < 0$	f is decreasing
$(-1, \infty)$	$(+)(+) > 0$	f is increasing

Function f is increasing on $(-\infty, -2) \cup (-1, \infty)$ and decreasing on $(-2, -1)$.

(d) $f(x) = 6 - 9x - x^2$ $\quad \Rightarrow f'(x) = -9 - 2x$

If $f'(x) = 0$

$\Rightarrow -9 - 2x = 0 \quad \Rightarrow x = -\frac{9}{2}$

$f'(x) = -9 - 2x > 0$, For all $x \in \left(-\infty, -\frac{9}{2}\right)$, therefore, the function is strictly increasing on $\left(-\infty, -\frac{9}{2}\right)$.

$f'(x) = -9 - 2x < 0$, For all $x \in \left(-\frac{9}{2}, \infty\right)$, therefore, the function is strictly decreasing on $\left(-\frac{9}{2}, \infty\right)$.

(e) $f(x) = (x+1)^3(x-3)^3$

$\Rightarrow f'(x) = 3(x+1)^2(x-3)^3 + (x+1)^3 . 3(x-3)^2$

$= 3(x+1)^2(x-3)^2(x - 3 + x + 1) = 3(x+1)^2(x-3)^2(2x-2) = 6(x+1)^2(x-3)^2(x-1)$

If $f'(x) = 0$

$\Rightarrow 6(x+1)^2(x-3)^2(x-1) = 0$

$\Rightarrow x = -1, 1, 3$

$x = -1, x = 1$ and $x = 3$ divides the number lines into four intervals $(-\infty, -1)$, $(-1, 1)$, $(1, 3)$ and $(3, \infty)$.

Intervals	Sign of $f'(x)$	Nature of f
$(-\infty, -1)$	$(+)(+)(-) < 0$	f is decreasing
$(-1, 1)$	$(+)(+)(-) < 0$	f is decreasing
$(1, 3)$	$(+)(+)(+) > 0$	f is increasing
$(3, \infty)$	$(+)(+)(+) > 0$	f is increasing

Function f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$.

Question 7:

Show that $y = \log(1+x) - \frac{2x}{2+x}, x > -1$, is an increasing function of x throughout its domain.

Answer 7:

$$y = \log(1+x) - \frac{2x}{2+x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x} - \frac{(2+x).2 - 2x(1)}{(2+x)^2} = \frac{1}{1+x} - \frac{4}{(2+x)^2} = \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4 + x^2 + 4x - 4 - 4x}{(1+x)(2+x)^2} = \frac{x^2}{(1+x)(2+x)^2}$$

$x^2 > 0$ and $(2+x)^2 > 0$, as these are perfect square and $(1+x) > 0$ as $x > -1$.

Therefore, $\frac{dy}{dx} > 0$, if $x > -1$. Hence, the function is increasing throughout its domain.

Question 8:

Find the values of x for which $y = [x(x-2)]^2$ is an increasing function.

Answer 8:

$y = [x(x-2)]^2 \quad \Rightarrow \frac{dy}{dx} = 2[x(x-2)].\frac{d}{dx}[x(x-2)] = 2[x(x-2)](2x-2) = 4[x(x-2)(x-1)]$

If $f'(x) = 0 \qquad \Rightarrow 4[x(x-2)(x-1)] = 0 \qquad \Rightarrow x = 0, 1, 2$

$x = 0, x = 1$ and $x = 2$ divides the number line into four intervals $(-\infty, 0), (0, 1), (1, 2)$ and $(2, \infty)$.

Intervals	**Sign of $y'(x)$**	**Nature of y**
$(-\infty, 0)$	$(-)(-)(-) < 0$	f is decreasing
$(0, 1)$	$(+)(-)(-) > 0$	f is increasing
$(1, 2)$	$(+)(-)(+) < 0$	f is decreasing
$(2, \infty)$	$(+)(+)(+) > 0$	f is increasing

Function y is increasing on $(0,1) \cup (2, \infty)$.

Question 9:

Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$, is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

Answer 9:

$$y = \frac{4\sin\theta}{(2 + \cos\theta)} - \theta$$

$$\Rightarrow \frac{dy}{d\theta} = \frac{(2+\cos\theta).4\cos\theta - 4\sin\theta(0 - \sin\theta)}{(2+\cos\theta)^2} - 1 = \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$\Rightarrow \frac{dy}{d\theta} = \frac{8\cos\theta + 4 - 4 - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2} = \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2} = \frac{\cos\theta\,(4 - \cos\theta)}{(2+\cos\theta)^2}$$

$(2+\cos\theta)^2 > 0$, as it is a perfect square. $\cos\theta > 0$ as $\theta \in \left[0, \frac{\pi}{2}\right]$.

$(4 - \cos\theta) > 0$, because $0 \le \cos\theta \le 1$, for all $\theta \in \left[0, \frac{\pi}{2}\right]$.

Therefore, $\frac{dy}{d\theta} > 0$, if $\theta \in \left[0, \frac{\pi}{2}\right]$.

Hence, y is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

Question 10:

Prove that the logarithmic function is strictly increasing on $(0, \infty)$.

Answer 10:

$f(x) = \log x \qquad \Rightarrow f'(x) = \frac{1}{x} > 0, \quad$ For all $x \in (0, \infty)$.

Therefore, the function is strictly increasing on $(0, \infty)$.

Question 11:

Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor strictly decreasing on (–1, 1).

Answer 11:

$f(x) = x^2 - x + 1 \Rightarrow f'(x) = 2x - 1$

If $f'(x) = 0 \quad \Rightarrow 2x - 1 = 0 \quad \Rightarrow x = \frac{1}{2}$

$x = \frac{1}{2}$ divides the interval $(-1, 1)$ into two intervals $\left(-1, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right)$.

$f'(x) = 2x - 1 < 0$, For all $x \in \left(-1, \frac{1}{2}\right)$, therefore, the function is strictly decreasing on $\left(-1, \frac{1}{2}\right)$.

$f'(x) = \frac{1}{x} > 0$, For all $x \in \left(\frac{1}{2}, 1\right)$, therefore, the function is strictly increasing on $\left(\frac{1}{2}, 1\right)$.

Therefore, the function is neither increasing nor decreasing on $(-1, 1)$.

Question 12:

Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$?

(A) cos x **(B)** cos 2x **(C)** cos 3x **(D)** tan x

Answer 12:

(A) $f(x) = \cos x \Rightarrow f'(x) = -\sin x$

$f'(x) = -\sin x < 0$, For all $x \in \left(0, \frac{\pi}{2}\right)$, therefore, the function is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

(B) $f(x) = \cos 2x \quad \Rightarrow f'(x) = -2\sin 2x$

If $x \in \left(0, \frac{\pi}{2}\right)$, then $2x \in (0, \pi)$, but $\sin 2x > 0$ if $2x \in (0, \pi)$. Therefore, $-\sin 2x < 0$ if $2x \in (0, \pi)$.

$f'(x) = -2\sin 2x < 0$, For all $x \in \left(0, \frac{\pi}{2}\right)$, therefore, the function is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

(C) $f(x) = \cos 3x \quad \Rightarrow f'(x) = -3\sin 3x$

If $x \in \left(0, \frac{\pi}{2}\right)$, then $3x \in \left(0, \frac{3\pi}{2}\right)$, but $\sin 3x > 0$ if $3x \in (0, \pi)$. So, $-\sin 3x < 0$ if $3x \in (0, \pi)$ or $x \in \left(0, \frac{\pi}{3}\right)$

$f'(x) = -2\sin 3x < 0$, For all $x \in \left(0, \frac{\pi}{3}\right)$, therefore, the function is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

$\sin 3x < 0$ if $3x \in \left(\pi, \frac{3\pi}{2}\right)$, therefore, $-\sin 3x > 0$ if $3x \in \left(\pi, \frac{3\pi}{2}\right)$ or $x \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

$f'(x) = -2\sin 3x > 0$, For all $x \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, therefore, the function is strictly increasing on $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

Hence, the function is neither increasing nor decreasing on $\left(0, \frac{\pi}{2}\right)$.

(D) $f(x) = \tan x \quad \Rightarrow f'(x) = \sec^2 x$

$f'(x) = \sec^2 x > 0$, For all $x \in \left(0, \frac{\pi}{2}\right)$, therefore, the function is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

Therefore, $\cos x$ and $\cos 2x$ are decreasing on $\left(0, \frac{\pi}{2}\right)$. Hence, the option (A) and (B) are correct.

Question 13:

On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ strictly decreasing?

(A) (0, 1) **(B)** $\left(\frac{\pi}{2}, \pi\right)$ **(C)** $\left(0, \frac{\pi}{2}\right)$ **(D)** None of these

Answer 13:

$$f(x) = x^{100} + sinx - 1 \quad \Rightarrow f'(x) = 100x^{99} + \cos x$$

(A) $100x^{99} > 0$ and $\cos x > 0$, For all $x \in (0, 1)$, therefore, the function is strictly increasing on $(0, 1)$.

(B) $100x^{99} > 0$ and $\cos x < 0$, For all $x \in \left(\frac{\pi}{2}, \pi\right)$, But $100x^{99} > \cos x$, for all $x \in \left(\frac{\pi}{2}, \pi\right)$.

Therefore, $f'(x) = 100x^{99} + \cos x > 0$, therefore, the function is strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$.

(C) $100x^{99} > 0$ and $\cos x > 0$, For all $x \in \left(0, \frac{\pi}{2}\right)$, therefore, the function is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

Hence, is function is not decreasing in the given intervals.

Hence, the option (D) is correct.

Question 14:

For what values of a the function f given by $f(x) = x^2 + ax + 1$ is increasing on $[1, 2]$?

Answer 14:

$f(x) = x^2 + ax + 1$

$\Rightarrow f'(x) = 2x + a$

But f is increasing on $(1, 2)$

$\Rightarrow f'(x) > 0 \quad \Rightarrow 2x + a > 0 \quad \Rightarrow x > -\frac{a}{2}$

Here, $x > -\frac{a}{2}$ when $x \in (1, 2)$ or $1 < x < 2$,

Therefore, for the least value of a, when the function is increasing on $(1, 2)$, we have

$-\frac{a}{2} = 1 \quad \Rightarrow a = -2$

Question 15:

Let I be any interval disjoint from (–1, 1). Prove that the function f given by $f(x) = x + \frac{1}{x}$ is strictly increasing on I.

Answer 15:

$$f(x) = x + \frac{1}{x} \qquad \Rightarrow f'(x) = x - \frac{1}{x^2} = \frac{x^2 - 1}{x^2} = \frac{(x-1)(x+1)}{x^2}$$

If $f'(x) = 0$

$\Rightarrow (x-1)(x+1) = 0$

$\Rightarrow x = -1, 1$

$x = -1$ and $x = 1$ divides the real value into intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.

Only $(-\infty, -1)$ and $(1, \infty)$ are disjoint from (– 1, 1).

Intervals	Sign of $f'(x)$	Nature of f
$(-\infty, -1)$	$(-)(-) > 0$	f is strictly increasing
$(1, \infty)$	$(+)(+)(+) > 0$	f is strictly increasing

Function f is increasing on $(-\infty, -1) \cup (1, \infty)$.

Hence, the function f is strictly increasing on interval I.

Question 16:

Prove that the function f given by $f(x) = \log \sin x$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Answer 16:

$$f(x) = \log \sin x \Rightarrow f'(x) = \frac{1}{\sin x} . \cos x = \tan x$$

$f'(x) = \tan x > 0$, for all $x \in \left(0, \frac{\pi}{2}\right)$, therefore, the function is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

$f'(x) = \tan x < 0$, for all $x \in \left(\frac{\pi}{2}, \pi\right)$, therefore, the function is strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Question 17:

Prove that the function f given by $f(x)=\log|\cos x|$ is decreasing on $\left(0,\frac{\pi}{2}\right)$ and increasing on $\left(\frac{3\pi}{2},2\pi\right)$.

Answer 17:

We have $f(x)=\log|\cos x|$

Therefore,

$$f'(x)=\frac{1}{\cos x}(-\sin x)$$
$$=-\tan x$$

In interval $\left(0,\frac{\pi}{2}\right)$, $\tan x>0 \Rightarrow -\tan x<0$

Hence, $f'(x)<0$

Thus, f is strictly decreasing on $\left(0,\frac{\pi}{2}\right)$.

In interval $\left(\frac{3\pi}{2},2\pi\right)$, $\tan x<0 \Rightarrow -\tan x>0$

Hence, $f'(x)>0$

Thus, f is strictly increasing on $\left(\frac{3\pi}{2},2\pi\right)$.

Question 18:

Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in **R**.

Answer 18:

$f(x) = x^3 - 3x^2 + 3x - 100$

$\Rightarrow f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$

$3(x-1)^2 \geq 0$, as it is perfect square.

Therefore, the function $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in **R**.

Question 19:

The interval in which $y = x^2 e^{-x}$ is increasing is

(A) $(-\infty, \infty)$ (B) $(-2, 0)$ (C) $(2, \infty)$ (D) $(0, 2)$

Answer 19:

$y = x^2 e^{-x}$

$\Rightarrow \dfrac{dy}{dx} = 2xe^{-x} - x^2 e^{-x} = xe^{-x}(2-x)$

If $f'(x) = 0$

$\Rightarrow xe^{-x}(2-x) = 0$

$\Rightarrow x = 0, 2$

$x = 0$ and $x = 2$ divides the real values into three intervals $(-\infty, 0)$, $(0, 2)$ and $(2, \infty)$.

Intervals	Sign of $y'(x)$	Nature of function y
$(-\infty, 0)$	$(-)(+) < 0$	f is strictly decreasing
$(0, 2)$	$(+)(+) > 0$	f is strictly increasing
$(2, \infty)$	$(+)(-) < 0$	f is strictly decreasing

Function y is strictly increasing on $(0,2)$.

Therefore, the option (D) is correct.

Mathematics

(Chapter – 6) (Application of Derivatives) (Exercise 6.3)
(Class – XII)

Question 1:

Find the maximum and minimum values, if any of the following functions given by.

(i) $f(x) = (2x+1)^2 + 3$ **(ii)** $f(x) = 9x^2 + 12x + 2$

(iii) $f(x) = -(x-1)^2 + 10$ **(iv)** $g(x) = x^3 + 1$

Answer 1:

(i) The given function is $f(x) = (2x+1)^2 + 3$

It can be observed that $(2x+1)^2 \geq 0$ for every $x \in R$,

Therefore, $f(x) = (2x+1)^2 + 3 \geq 3$ for every $x \in R$

The minimum value of f is attained when 2x+1 = 0

2x+1 = 0 $\therefore x = -\frac{1}{2}$

Minimum value of f = $f\left(-\frac{1}{2}\right) = \left[2.\left(-\frac{1}{2}\right)+1\right]^2 + 3 = 3$

Hence, function f does not have a maximum value.

(ii) The given function is $f(x) = 9x^2 + 12x + 2 = (3x+2)^2 - 2$.

It can be observed that $(3x+2)^2 \geq 0$ for every $x \in R$.

Therefore, $f(x) = (3x+2)^2 - 2 \geq 3$ for every $x \in R$

The minimum value of f is attained when 3x+2 = 0

3x + 2 = 0 $\therefore x = \frac{-2}{3}$

$\therefore$ Minimum value of f = $f\left(\frac{-2}{3}\right) = \left(3.\left(\frac{-2}{3}\right)+2\right)^2 - 2 = -2$

Hence, function f does not have a maximum value.

(iii) The given function is $f(x) = -(x-1)^2 + 10$

It can be observed that $(x-1)^2 \geq 0$ for every $x \in R$.

Therefore, $f(x) = -(x-1)^2 + 10 \leq 10$ for every $x \in R$

The maximum value of f is attained when (x-1) = 0

$(x-1) = 0 \quad \therefore x = 1$

Maximum value of $f = f(1) = -(1-1)^2 + 10 = 10$

Hence, function f does not have a minimum value.

(iv) The given function is $g(x) = x^3 + 1$

Here, $x^3 > 0$ if $x > 0$ and $x^3 < 0$ if $x < 0$.

Hence, function g neither has a maximum value nor a minimum value.

Question 2:

Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = |x+2| - 1$ **(ii)** $g(x) = -|x+1| + 3$ **(iii)** $h(x) = \sin(2x) + 5$

(iv) $f(x) = |\sin 4x + 3|$ **(v)** $h(x) = x + 1, x \in (-1, 1)$

Answer 2:

(i) $f(x) = |x+2| - 1$

We know that $|x+2| \geq 0$ for every $x \in R$, therefore, f(x) = $|x+2| - 1 \geq -1$ for every $x \in R$

The minimum value of f is attained when. $|x+2| = 0$

$|x+2| = 0 \quad \Rightarrow x = -2$

$\therefore$ Minimum value of f = f (-2) = $|-2+2| - 1 = -1$

Hence, function f does not have a maximum value.

(ii) $g(x) = -|x+1| + 3$

We know that $-|x+1| \leq 0$ for every $x \in R$

Therefore, f(x) = $-|x+1| + 3 \leq 3$ for every $x \in R$

The maximum value of g is attained when $|x+1| = 0$

$|x+1| = 0 \quad \Rightarrow x = -1$

$\therefore$ Maximum value of $g = g(-1) = -|-1+1| + 3 = 3$

Hence, function g does not have a minimum value.

(iii) $h(x) = \sin(2x) + 5$

We now that $-1 \leq \sin 2x \leq 1$

$\therefore -1 + 5 \leq \sin 2x + 5 \leq 1 \leq 1 + 5$

$\therefore 4 \leq \sin 2x + 5 \leq 6$

Hence, the maximum and minimum values of h are 6 and 4 respectively.

(iv) $f(x) = |\sin 4x + 3|$

We know that $-1 \leq \sin 4x \leq 1$

$\therefore 2 \leq \sin 4x + 3 \leq 4$

$\therefore 2 \leq |\sin 4x + 3| \leq 4$

Hence, the maximum and minimum values of f are 4 and 2 respectively.

(v) The given function is h (x) = x + 1, x ∈ (- 1, 1)

Here, if a point x_0 is closest to - 1, then we find $x_0/2 + 1 > x_0 + 1$ for all $x_0 \in$ (- 1, 1).

Also, if x_1 is closest to 1, then find$(x_1 + 1) < (x_1 + 1)/2 + 1$ for all $x_1 \in$ (- 1, 1).

Hence, function h (x) has neither maximum nor minimum value in (- 1, 1)

Question 3:

Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i) $f(x) = x^2$

(ii) $g(x) = x^3 - 3x$

(iii) $h(x) = \sin x + \cos 0 \quad < x < \frac{\pi}{2}$

(iv) $f(x) \ \sin x - \cos x,\ 0 < x < 2\pi$

(v) $f(x) = x^3 - 6x^2 + 9x + 15$

(vi) $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

(vii) $g(x) = \frac{1}{x^2+2}$

(viii) $f(x) = x\sqrt{1-x},\ 0 < x < 1$

Answer 3:

(i) $f(x) = x^2$

$f'(x) = 2x$

Now, $f'(x) = 0 \quad \Rightarrow x = 0$

Thus, x = 0 is the only critical point which could possibly be the point of local maxima or local minima of f.

We have $f''(0) = 0$, which is positive.

Therefore, by second derivative test, $x = 0$ is a point of local minima and local minimum value of f at $x = 0$ is $f\ (0) = 0$.

(ii) $g(x) = x^3 - 3x$

$g'(x) = 3x^2 - 3$

Now, $g'(x) = 0 \ \Rightarrow x = \pm 1$ and $g''(x) = 6x$

$g''(1) = 6 > 0$ and $g''(-1) = -6 > 0$

By second derivative test, $x = 1$ is a point of local minima and local minimum value of g at $x = 1$ is $g(1) = 1^3 - 3 = 1 - 3 = -2$
However, x = −1 is a point of local maxima and local maximum value of g at x = −1 is
$g(-1) = (-1)^3 - 3(-1) = -1 + 3 = 2$

(iii) $h(x) = \sin x + \cos 0 \quad < x < \frac{\pi}{2}$
$h'(x) = \cos x - \sin x$
$h'(x) = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$
$h''(x) = -\sin x - \cos x = -(\sin x + \cos x)$
$h''\left(\frac{\pi}{4}\right) = -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}} = -\sqrt{2} < 0$
Therefore, by second derivative test, $x = \frac{\pi}{4}$ is a point of local maxima and the local maximum value of h at $x = \frac{\pi}{4}$ is $h\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$
(iv) $f(x)\ \sin x - \cos x,\ 0 < x < 2\pi$
$f'(x) = \cos x + \sin x$
$f'(x) = 0 \Rightarrow \cos x = -\sin x \Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0, 2\pi)$
$f''(x) = -\sin x + \cos x$
$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} > 0$
$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$
Therefore, by second derivative test, $x = \frac{3\pi}{4}$ is a point of local maxima and the local maximum value of f at $x = \frac{3\pi}{4}$ is $f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$
However, $x = \frac{7\pi}{4}$ is a point of local minima and the local minimum value of f is
$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$

(v) $f(x) = x^3 - 6x^2 + 9x + 15$
$f'(x) = 3x^2 - 12x + 9$
$f'(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0 \Rightarrow 3(x-1)(x-3) = 0 \quad \Rightarrow x = 1, 3$
Now, $f''(x) = 6x - 12 = 6(x-2)$
$f''(1) = 6(1-2) = -6 < 0$ and $f''(3) = 6(3-2) = 6 > 0$
Therefore, by second derivative test, x = 1 is a point of local maxima and the local maximum value of f at x=1 is $f(1) = 1 - 6 + 9 + 15 = 19$
However, x = 3 is a point of local minima and the local minimum value of f at x = 3 is
$f(3) = 27 - 54 + 27 + 15 = 15$

(vi) $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$
$\therefore g'(x) = \frac{1}{2} - \frac{2}{x^2}$
Now, $g'(x) = 0$ gives $\frac{2}{x^2} = \frac{1}{2} \Rightarrow x^2 = 4 \Rightarrow x \pm 2$
Since x>0, we take x = 2.
Now $g''(x) = \frac{4}{x^3}$, therefore $g''(2) = \frac{4}{(2)^3} = \frac{1}{2} > 0$
Therefore, by second derivative test, $x = 2$ is a point of local minima and the local minimum value of g at $x = 2$ is $g(2) = \frac{2}{2} + \frac{2}{2} = 1 + 1 = 2$

(vii) $g(x) = \frac{1}{x^2+2}$

$g'(x) = \frac{-(2x)}{(x^2+2)^2}$

$g'(x) = 0 \quad \Rightarrow \frac{-(2x)}{(x^2+2)^2} \quad = 0 \Rightarrow x = 0$

Now, for values close to x = 0 and to the left of 0, $g'(x) > 0$ Also, for values close to $x = 0$ and to the right of 0, $g'(x) < 0$.

Therefore, by first derivative test, $x = 0$ is a point of local maxima and the local maximum value of $g(0)$ *is* $\frac{1}{0+2} = \frac{1}{2}$

(viii) $f(x) = x\sqrt{1-x},\ \ 0 < x < 1$

Therefore,

f' (x) = $\sqrt{1 - x}$ + (1/(2$\sqrt{1 - x}$)) (- 1)

= $\sqrt{1 - x}$ + x/(2$\sqrt{1 - x}$)

= (2 (1 - x) - x)/(2$\sqrt{1 - x}$)

= (2 - 3x)/(2$\sqrt{1 - x}$)

Now,

f' (x) = 0

= (2 - 3x)/(2$\sqrt{1 - x}$) = 0

⇒ 2 - 3x = 0

⇒ x = 2/3

Also,

f'' (x) = 1/2 [($\sqrt{1 - x}$ (- 3) - (2 - 3x) (- 1/2 $\sqrt{1 - x}$)) / (1 - x)]

= [($\sqrt{1 - x}$ (- 3) - (2 - 3x) (- 1/2$\sqrt{1 - x}$)) / 2 (1 - x)]

= = [- 6(1- x) + (2 - 3x)] / [4 (1 - x) ($\sqrt{1 - x}$)]

$= (3x - 4)/4 (1 - x)^{3/2}$

Hence,

$f''(2/3) = [3(2/3) - 4] / [4 (1 - 2/3)^{3/2}]$

$= (2 - 4) / 4 (1 - 2/3)^{3/2}$

$= - 1/2 (1/3)^{3/2} < 0$

Therefore, by second derivative test, x = 2/3 is a point of local maxima and the local maximum value of f at x = 2/3

$f(2/3) = 2/3 \sqrt{1 - 2/3}$

$= 2/3 \sqrt{1/3}$

$= 2/3\sqrt{3}$

$= 2\sqrt{3}/9$

Question 4:

Prove that the following functions do not have maxima or minima:

(i) $f(x) = e^x$ **(ii)** $g(x) = \log x$ **(iii)** $h(x) = x^3 + x^2 + x + 1$

Answer 4:

(i) We have,

$f(x) = e^x$

$\therefore f'(x) = e^x$

Now if $f'(x) = 0$, then $e^x = 0$

But, the exponential function can never assume 0 for any value of x.

Therefore, there does not exist c ∈ R such that $f'(c) = 0$

Hence, function f does not have maxima or minima.

(ii) We have, $g(x) = log\, x$

$g'(x) = \frac{1}{x}$, since log x is defined for a positive number x, $g'(x) > 0$ for any x.

Therefore, there does not exist c ∈ R such that $g'(c) = 0$

Hence, function g does not have maxima or minima.

(iii) We have $h(x) = x^3 + x^2 + x + 1$

$h'(x) = 3x^2 + 2x + 1$

Now, $h(x) = 0$ $3x^2 + 2x + 1 = 0$ $x = \frac{-2\pm2\sqrt{2}\,i}{6} = \frac{-1\pm\sqrt{2}\,i}{3} \notin R$

Therefore, there does not exist c ∈ R such that. $h'(c) = 0$

Hence, function h does not have maxima or minima.

Question 5:

Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $f(x) = x^3, x \in [-2,2]$

(ii) $f(x) = \sin x + \cos x, x \in [0,\pi]$

(iii) $f(x) = 4x - \frac{1}{2}x^2,\ x \in \left[-2,\frac{9}{2}\right]$

(iv) $f(x) = (x-1)^2 + 3,\ x \in [-3,1]$

Answer 5:

(i) The given function is $f(x) = x^3$

$f'(x) = 3x^2$

Now, $f'(x) = 0 \Rightarrow x = 0$

Then, we evaluate the value of f at critical point x = 0 and at end points of the interval $[-2, 2]$.

$f(0) = 0,\ f(-2) = (-2)^3 = -8,\quad f(2) = (2)^3 = 8$

So, we can conclude that the absolute maximum value of f on $[-2, 2]$ is 8 at x = 2.

Also the absolute minimum value of f on $[-2, 2]$ is -8 occurring at x = -2.

(ii) The given function is $f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

Now, $f'(x) = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$

Then, we evaluate the value of f at critical point $x = \frac{\pi}{4}$ and at the end points of the interval $[0, \pi]$.

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + cos\,\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$f(0) = \sin 0 + cos\ 0 = 0 + 1 = 1$

$f(\pi) = \sin\pi + cos\ \pi = 0 - 1 = -1$

Hence, we can conclude that the absolute maximum value of f on $[0,\pi]$ is $\sqrt{2}$ occurring at $x = \frac{\pi}{4}$ and the absolute minimum value of f on $[0,\pi]$ is -1 occurring at $x = \pi$.

(iii) The given function is $f(x) = 4x - \frac{1}{2}x^2$

$f'(x) = 4 - \frac{1}{2}(2x) = 4 - x$

Now, $f'(x) = 0 \quad \Rightarrow x = 4$

Then, we evaluate the value of f at critical point x = 4 and at the end points of the interval $\left[-2,\frac{9}{2}\right]$.

$f(4) = 16 - \frac{1}{2}(16) = 16 - 8 = 8$

$f(-2) = -2 - \frac{1}{2}(4) = -8 - 2 = -10$

$f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2 = 18 - \frac{81}{8} = 18 - 10.125 = 7.875$

Hence, we can conclude that the absolute maximum value of f on $\left[-2, \frac{9}{2}\right]$ is 8 occurring at x = 4 and the absolute minimum value of f on $\left[-2, \frac{9}{2}\right]$ is -10 occurring at x = − 2.

(iv) The given function is $f(x) = (x-1)^2 + 3$

$f'(x) = 2(x-1)$

Now, $f'(x) = 0 \Rightarrow 2(x-1) = 0 \therefore x = 1$

Then, we evaluate the value of f at critical point x = 1 and at the end points of the interval $[-3, 1]$

$f(1) = (1-1)^2 + 3 = 0 + 3 = 3$

$f(-3) = (-3-1)^2 + 3 = 16 + 3 = 19$

Hence, we can conclude that the absolute maximum value of f on $[-3, 1]$ is 19 occurring at x = -3 and the minimum value of f on $[-3, 1]$ is 3 occurring at x = 1.

Question 6:

Find the maximum profit that a company can make, if the profit function is given by

$p(x) = 41 - 72x - 18x^2$

Answer 6:

The profit function is given as $p(x) = 41 - 72x - 18x^2$.

∴ p'(x)=-72-36x

⇒x=-7236=-2

Also,

p''(-2)=-36<0

By second derivative test, x=-2 is the point of local maxima of *p*.

∴ Maximum profit=p-2

=41-72(-2)-18(-2)2=41+144-72=113

Hence, the maximum profit that the company can make is 113 units.

Question 7:

Find both the maximum value and the minimum value of $3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval [0, 3].

Answer 7:

Let $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$

$f'(x) = 12x^3 - 24x^2 + 24x - 48 = 12(x^3 - 2x^2 + 2x - 4) = 12[x^2(x-2) + 2(x-2)]$

$= 12(x-2)(x^2+2)$

Now, $f'(x) = 0$ gives x =2 or $x^2 + 2 = 0$ for which there are real roots

Therefore, we consider only $x = 2 \in [0, 3]$

Now, we evaluate the value of f at critical point x = 2 and at the end points of the interval [0, 3].

$f(2) = 3(2)^4 - 8(2)^3 + 12(2)^2 - 48(2) + 25$
$= 3(16) - 8(8) + 12(4) - 48(2) + 25 = 48 - 64 + 48 - 96 + 25 = -39$
$f(0) = 3(0)^4 - 8(0)^3 + 12(0)^2 - 48(0) + 25 = 3(0) - 8(0) + 12(0) - 48(0) + 25 = 25$
$f(3) = 3(3)^4 - 8(3)^3 + 12(3)^2 - 48(3) + 25$ $F(3) = 3(81) - 8(27) + 12(9) - 48(3) + 25$
$= 243 - 216 + 108 - 144 + 25 = 16$
Hence, we can conclude that the absolute value of f on [0, 3] is 25 occurring at $x = 0$ and the absolute minimum value of f at [0, 3] is -39 occurring at $x = 2$.

Question 8:
At what points in the interval $[0, 2\pi]$ does the function sin 2x attain its maximum value?
Answer 8:
Let $f(x) = sin\, 2x \Rightarrow f'(x) = 2\cos 2x$
Now, $f'(x) = 0 \Rightarrow \cos 2x = 0$

$$\Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \qquad \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Then, we evaluate the values of f at critical points $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and at the end points of the interval $[0, 2\pi]$.

$$f\left(\frac{\pi}{4}\right) = sin\, \frac{\pi}{2} = 1,\ f\left(\frac{3\pi}{4}\right) = sin\left(\frac{3\pi}{4}\right) = -1$$

$$f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) = 1,\ f\left(\frac{7\pi}{4}\right) = \sin\left(\frac{7\pi}{4}\right) = -1$$

$$f(0) = \sin(0) = 0\ , f(2\pi) = \sin(2\pi) = 0$$

Hence, we can conclude that the absolute maximum value of f on $[0, 2\pi]$ is occurring at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Question 9:
What is the maximum value of the function $\sin x + \cos x$?
Answer 9:

Let $f(x) = \sin x + \cos x \Rightarrow f'(x) = \cos x - \sin x$

$$f'(x) = 0 \quad \Rightarrow \sin x = \cos x \qquad \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$$

$$f''(x) = -\sin x - \cos x = -(\sin x + \cos x)$$

Now, $f''(x)$ will be negative when $(\sin x + \cos x)$ is positive i.e., when sin x and cos x are both positive. Also, we know that sin x and cos x both are positive in the first quadrant. Then we consider $x = \frac{\pi}{4}$

$$f''\left(\frac{\pi}{4}\right) = -\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) = -\left(\frac{2}{\sqrt{2}}\right) = -\sqrt{2} < 0$$

$\therefore$By second derivative test, f will be the maximum at $x = \frac{\pi}{4}$ and the maximum value of f is

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Question 10:
Find the maximum value of $2x^3 - 24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in $[-3, -1]$.

Answer 10:
Let $f(x) = 2x^3 - 24x + 107$
$f'(x) = 6x^2 - 24 = 6(x^2 - 4)$
Now, $f'(x) = 0 \Rightarrow 6(x^2 - 4) = 0 \Rightarrow x^2 = 4 \Rightarrow x \pm 2$
We first consider the interval [1,3]
Then, we evaluate the value of f at the critical point $x = 2 \in [1,3]$ and at the end points of the interval [1,3].
$f(2) = 2(8) - 24(2) + 107 = 16 - 48 + 107 = 75$
$f(1) = 2(1) - 24(1) + 107 = 2 - 24 + 107 = 85$
$f(3) = 2(27) - 24(3) + 107 = 54 - 72 + 107 = 89$
Hence, the absolute maximum value of $f(x)$ in the interval [1, 3] is 89 occurring at $x = 3$.
Next, we consider the interval [−3, −1].
Evaluate the value of f at the critical point x = −2 ∴ [−3, −1] and at the end points of the interval [1, 3].
$f(-3) = 2(-27) - 24(-3) + 107 = -54 + 72 + 107 = 125$
$f(-1) = 2(-1) - 24(-1) + 107 = -2 + 24 + 107 = 129$
$f(-2) = 2(-8) - 24(-2) + 107 = -16 + 48 + 107 = 139$
Hence, the absolute maximum value of $f(x)$ in the interval$[-3, -1]$ is 139 occurring at $x = -2$.

Question 11:
It is given that at x = 1, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval $[0, 2]$ find the value of a.

Answer 11:
Let $f(x) = x^4 - 62x^2 + ax + 9$, $\therefore f'(x) = 4x^3 - 124x + a$
It is given that function f attains its maximum value on the interval [0,2] at x = 1.
$f'(1) = 0 \Rightarrow 4(1)^3 - 124(1) + a = 0 \quad \Rightarrow 4 - 124 + a = 0 \quad \Rightarrow a = 120$
Hence, the value of a is 120.

Question 12:
Find the maximum and minimum values of $x + \sin 2x$ $on[0, 2\pi]$.

Answer 12:

$$Let\ f(x) = x + \sin 2x$$

$f'(x) = 1 + 2\cos 2x$
Now, $f'(x) = 0 \Rightarrow \cos 2x = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$$\Rightarrow 2x = 2\pi, \pm\frac{2\pi}{3}, n \in Z \qquad \Rightarrow x = 2\pi \pm \frac{\pi}{3}, n \in Z \qquad \Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \in [0, 2\pi]$$

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

$$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin\frac{8\pi}{3} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$
$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin\frac{10\pi}{3} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}$$
$$f(0) = 0 + \sin 0 = 0$$
$$f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi$$
Hence, we can conclude that the absolute maximum value of $f(x)$ in the interval $[0,\ 2\pi]$ is 2π occurring at $x = 2\pi$ and the absolute minimum value of $f(x)$ in the interval $[0,\ 2\pi]$ is 0 occurring at $x = 0$.

Question 13:
Find two numbers whose sum is 24 and whose product is as large as possible.

Answer 13:
Let one number be x. Then, the other number is (24 – x).
Let $p(x)$ denote the product of the two numbers. Thus, we have
$p(x) = x(24 - x) = 24x - x^2$
$p'(x) = 24 - 2x$ and $p''(x) = -2$
Now, $p'(x) = 0 \Rightarrow x = 12$ also $p''(x) = -2 < 0$
∴ By second derivative test, $x = 12$ is the point of local maxima of p.
Hence, the product of the numbers is the maximum when the numbers are 12 and 24 – 12 = 12.

Question 14:
Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

Answer 14:
The two numbers are x and y such that $x + y = 60$ so $y = 60 - x$
Let $f(x) = xy^3 \Rightarrow f(x) = x(60 - x)^3$
$\therefore f'(x) = (60 - x)^3 - 3x(60 - x)^2 = (60 - x)^2[60 - x - 3x] = (60 - x)^2(60 - 4x)$
Now, $f'(x) = 0 \Rightarrow x = 60$ or $x = 15$
Now $f''(x) = -2(60 - x)(60 - 4x) - 4(60 - x)^2 = -2(60 - x)[60 - 4x + 2(60 - x)]$
When, x = 15, $f''(x) = -12(60 - 15)(30 - 15) = -12 \times 45 \times 15 < 0$.
When x = 60, $f''(x) = 0$
∴ By second derivative test, x = 15 is a point of local maxima of f.
Thus, function xy^3 is maximum when x = 15 and y = 60 – 15 = 45.
Hence, the required numbers are 15 and 45.

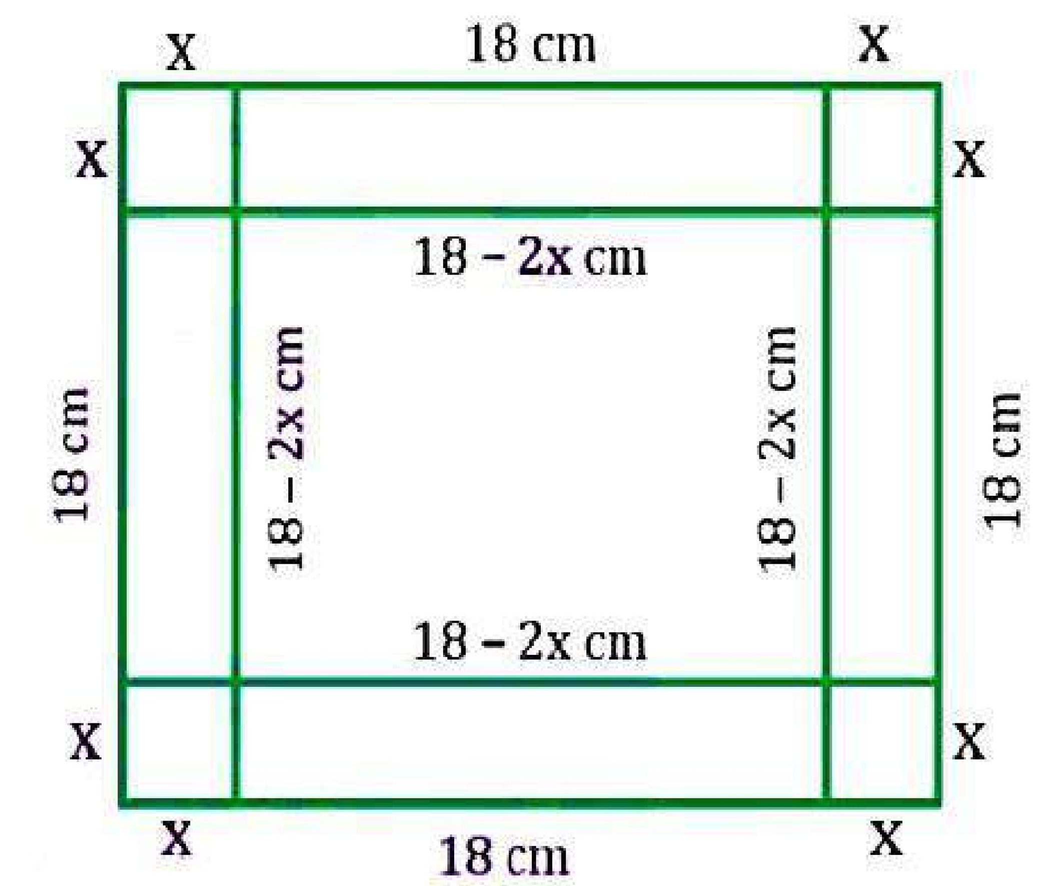
Question 15:
Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

Answer 15:
Let one number be x. Then, the other number is $y = 35 - x$.
Let $P(x) = x^2y^5$, then we have $P(x) = x^2(35 - x)^5$
$\Rightarrow P'(x) = 2x(35 - x)^5 - 5x^2(35 - x)^4 = x(35 - x)^4[2(35 - x) - 5x] = x(35 - x)^4(70 - 7x)$
$= 7x(35 - x)^4(10 - x)$
Now, $P'(x) = 0 \Rightarrow x = 0, 35, 10$

$P''(x) = 7(35-x)^4(10-x) + 7x[-(35-x)^4 - 4(35-x)^3(10-x)]$
$= 7(35-x)^4(10-x) - 7x(35-x)^4 - 28x(35-x)^3(10-x)$
$= 7(35-x)^3[(35-x)(10-x) - x(35-x) - 4x(10-x)]$
$= 7(35-x)^3[350 - 45x + x^2 - 35x + x^2 - 40x + 4x^2] = 7(35-x)^3(6x^2 - 120x + 350)$
When x = 35, y = 35 – 35 = 0 and the product x^2y^5 will be equal to 0.
When x = 0, y = 35 – 0 = 35 and the product x^2y^5 will be equal to 0
∴ x = 0 and x = 35 cannot be the possible values of x.
When x = 10, we have, $P''(x) = 7(35-x)^3(6\times 100 - 120\times 10 + 350) = 7(25)^3(-250) < 0$
∴ By second derivative test, $P(x)$ will be the maximum when $x = 10$ and $y = 35 - 10 = 25$.
Hence, the required numbers are 10 and 25.

Question 16:
Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.
Answer 16:
Let one number be x. Then, the other number is (16 – x).
Let the sum of the cubes of these numbers be denoted by $S(x)$. Then,
$\Rightarrow S(x) = x^3 + (16-x)^3 \therefore S'(x) = 3x^2 - 3(16-x)^2,\ S''(x) = 6x + 6(16-x)$
Now, $S'(x) = 0 \Rightarrow 3x^2 - 3(16-x)^2 = 0$
$\Rightarrow x^2 - (16-x)^2 = 0 \Rightarrow x^2 - 256 - x^2 + 32x = 0 \quad \Rightarrow x = \dfrac{256}{32} = 8$
Now $S''(8) = 6(8) + 6(16-8) = 48 + 48 = 96 > 0$
∴ By second derivative test, x = 8 is the point of local minima of S.
Hence, the sum of the cubes of the numbers is the minimum when the numbers are 8 and 16 – 8 = 8.

Question 17:
A square piece of tin of side 18 cm is to made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?
Answer 17:
Let the side of the square to be cut off be x cm. Then, the length and the breadth of the box will be (18 – 2x) cm each and the height of the box is x cm.

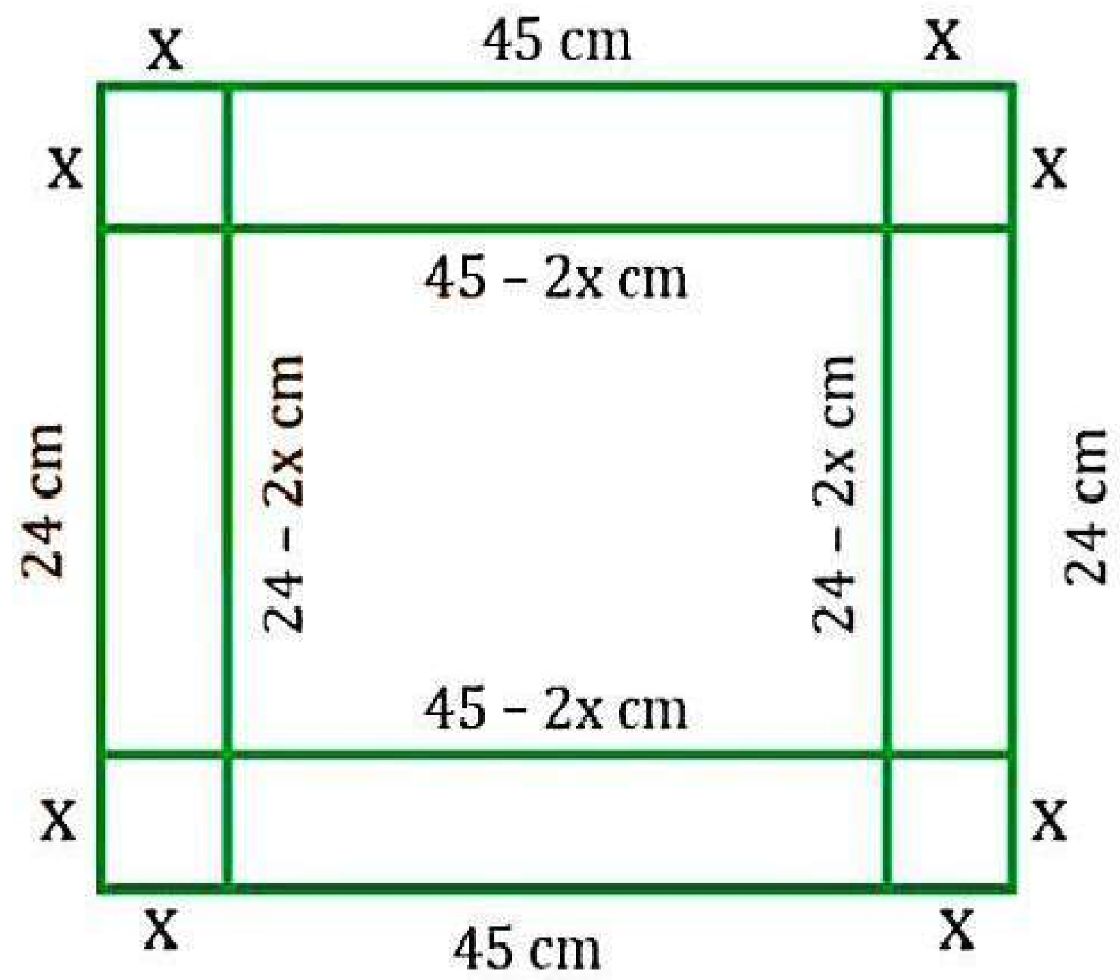


Therefore, the volume V(x) of the box is given by,
$V(x) = x(18-2x)^2$
$\Rightarrow V'(x) = (18-2x)^2 - 4x(18-2x) = (18-2x)[18-2x-4x]$
$= (18-2x)(18-6x) = 6\times 2(9-x)(3-x)$
$V''(x) = 12[-(9-x) - (3-x)]$
$= -12(9-x+3-x) = -12(12-2x) = -24(6-x)$
Now, $V'(x) = 0 \Rightarrow x = 9$ or $x = 3$
If x = 9, then the length and the breadth will become 0. Therefore $x \neq 9$ so $x = 3$.
Now, $V''(x) = -24(6-3) = -72 < 0$
By second derivative test, x = 3 is the point of maxima of V.
Hence, if we remove a square of side 3 cm from each corner of the square tin and make a box from the remaining sheet, then the volume of the box obtained is the largest possible.

Question 18:

A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?

Answer 18:



Let the side of the square to be cut off be x cm. Then, the height of the box is x, the length is 45 – 2x, and the breadth is 24 – 2x.

Therefore, the volume V(x) of the box is given by,

$V(x) = x(45 - 2x)(24 - 2x) \ = x(1080 - 90x - 48x + 4x^2)$

$= 4x^3 - 138x^2 + 1080x$

$\Rightarrow V'(x) = 12x^2 - 276x + 1080$

$= 12(x^2 - 23x + 90) = 12(x - 18)(x - 5)$

$V''(x) = 24x - 276 = 12(2x - 23)$

Now, $V'(x) = 0 \Rightarrow x = 18 \text{ or } x = 5$

It is not possible to cut off a square of side 18 cm from each corner of the rectangular sheet. Thus, x cannot be equal to 18.

∴ x = 5

Now, $V''(5) = 12(10 - 23) = 12(-13) = -156 < 0$

By second derivative test, x = 5 is the point of maxima.

Hence, the side of the square to be cut off to make the volume of the box maximum possible is 5 cm.

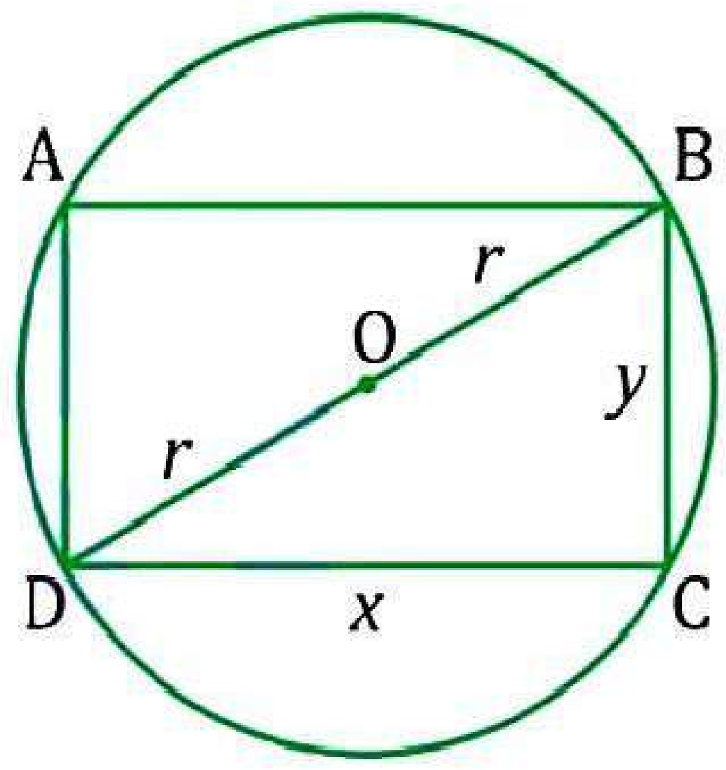
Question 19:

Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

Answer 19:

Let a rectangle of length l and breadth b be inscribed in the given circle of radius a. Then, the diagonal passes through the centre and is of length 2a cm.

Now, by applying the Pythagoras theorem, we have:



$(2a)^2 = (l)^2 + (b)^2$

$\Rightarrow b^2 = 4a^2 - l^2 \quad \Rightarrow b = \sqrt{4a^2 - l^2}$

Area of rectangle = $A = l\sqrt{4a^2 - l^2}$

$$\frac{dA}{dl} = \sqrt{4a^2 - l^2} + l\frac{1}{2\sqrt{4a^2 - l^2}}(-2l) = \sqrt{4a^2 - l^2} - \frac{l^2}{2\sqrt{4a^2 - l^2}}$$

$$= \frac{4a^2 - 2l^2}{\sqrt{4a^2 - l^2}}$$

$$\frac{d^2A}{dl^2} = \frac{\sqrt{4a^2 - l^2}(-4l) - (4a^2 - 2l^2)\frac{2l}{2\sqrt{4a^2 - l^2}}}{(4a^2 - l^2)} = \frac{(4a^2 - l^2)(-4l) + l(4a^2 - 2l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$= \frac{-12a^2 l - 2l^3}{(4a^2 - l^2)^{\frac{3}{2}}} = \frac{-2l(6a^2 - l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

Now, $\frac{dA}{dl} = 0$ gives $4a^2 = 2l^2 \ \Rightarrow l = \sqrt{2}a$

$\Rightarrow b = \sqrt{4a^2 - 2l^2} = \sqrt{2a^2} = \sqrt{2}\, a$

Now, when $l = \sqrt{2}\, a$

$$\frac{d^2A}{dl^2} = \frac{-2(\sqrt{2}\, a)(6a^2 - 2a^2)}{2\sqrt{2}\, a^3} = \frac{-8\sqrt{2}\, a^3}{2\sqrt{2}\, a^3} = -4 < 0$$

By the second derivative test, when $l = \sqrt{2}\, a$, then the area of the rectangle is the maximum.

Since $l = b = \sqrt{2}\, a$ the rectangle is a square.

Hence, it has been proved that of all the rectangles inscribed in the given fixed circle, the square has the maximum area.

Question 20:

Show that the right circular cylinder of given surface and maximum volume is such that is heights is equal to the diameter of the base.

Answer 20:

Let r and h be the radius and height of the cylinder respectively.

Then, the surface area (S) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi rh \quad \Rightarrow h = \frac{S - 2\pi r^2}{2\pi r} = \frac{S}{2\pi}\left(\frac{1}{r}\right) - r$$

Let V be the volume of the cylinder. Then,

$$V = \pi r^2 h = \pi r^2 \left[\frac{S}{2\pi}\left(\frac{1}{r}\right) - r \quad\right] = \frac{Sr}{2} - \pi r^3$$

Then, $\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2, \frac{d^2V}{dr^2} = -6\pi r$

Now, $\frac{dV}{dr} = 0 \Rightarrow \frac{S}{2} = 3\pi r^2 \Rightarrow r^2 = \frac{S}{6\pi}$

When $r^2 = \frac{S}{6\pi}$, $then$ $\frac{d^2V}{dr^2} = -6\pi r\left(\sqrt{\frac{S}{6\pi}}\right) < 0$

$\therefore$ By second derivative test, the volume is the maximum when $r^2 = \frac{S}{6\pi}$.

Now, when $r^2 = \frac{S}{6\pi}$ then $h = \frac{6\pi r^2}{2\pi}\left(\frac{1}{r}\right) - r = 3r - r = 2r$

Hence, the volume is the maximum when the height is twice the radius i.e., when the height is equal to the diameter.

Question 21:

Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimeters, find the dimensions of the can which has the minimum surface area?

Answer 21:

Let r and h be the radius and height of the cylinder respectively.

Then, volume (V) of the cylinder is given by, $V = \pi r^2 h = 100$ (given) so $h = \frac{100}{\pi r^2}$

Surface area (S) of the cylinder is given by, $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \frac{200}{r}$

$$\therefore \frac{dS}{dr} = 4\pi r - \frac{200}{r^2}, \frac{d^2S}{dr^2} = 4\pi + \frac{200}{r^3}$$

$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r = \frac{200}{r^2}$$

$$r^3 = \frac{200}{4\pi} = \frac{50}{\pi} \quad \text{so} \quad r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

Now, it is observed that when $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}, \frac{d^2S}{dr^2} > 0.$

$\therefore$By second derivative test, the surface area is the minimum when the radius of the cylinder is $\left(\frac{50}{\pi}\right)^{\frac{1}{3}} cm$

When $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, $\quad h = \dfrac{100}{\pi\left(\frac{50}{\pi}\right)^{\frac{2}{3}}} = \dfrac{2 \times 50}{(50)^{\frac{2}{3}}(\pi)^{1-\frac{1}{3}}} = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}} cm$

Hence, the required dimensions of the can which has the minimum surface area is given by radius$= \left(\frac{50}{\pi}\right)^{\frac{1}{3}} cm$ and height $=2\left(\frac{50}{\pi}\right)^{\frac{1}{3}} cm.$

Question 22:

A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Answer 22:

Let a piece of length l be cut from the given wire to make a square.

Then, the other piece of wire to be made into a circle is of length (28 – l) m.

Now, side of square = $\frac{l}{4}$

Let r be the radius of the circle. Then $2\pi r = 28 - l \Rightarrow r = \frac{1}{2\pi}(28 - l)$

The combined areas of the square and the circle (A) is given by,

$$A = (\text{side of the square})^2 + \pi r^2 = \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28 - l)\right]^2 = \frac{l^2}{16} + \frac{1}{4\pi}(28 - l)^2$$

$$\therefore \frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28 - l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28 - l)$$

$$\frac{d^2A}{dl^2} = \frac{1}{8} + \frac{1}{2\pi} > 0$$

Now, $\frac{dA}{dl} = 0 \Rightarrow \frac{l}{8} - \frac{1}{2\pi}(28 - l) = 0$

$$\Rightarrow \frac{\pi l - 4(28 - l)}{8\pi} = 0$$

$$\Rightarrow (\pi + 4)l - 112 = 0 \qquad \Rightarrow l = \frac{112}{\pi + 4}$$

Thus, when $l = \frac{112}{\pi+4}, \frac{d^2A}{dl^2} > 0$

$\therefore$ By second derivative test, the area (A) is the minimum when $l = \frac{112}{\pi+4}$.

Hence, the combined area is the minimum when the length of the wire in making the square is $\frac{112}{\pi+4}$ cm while the length of the wire in making the circle is $28 - \frac{112}{\pi+4} = \frac{28\pi}{\pi+4}\ cm.$

Question 23:

Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Answer 23:

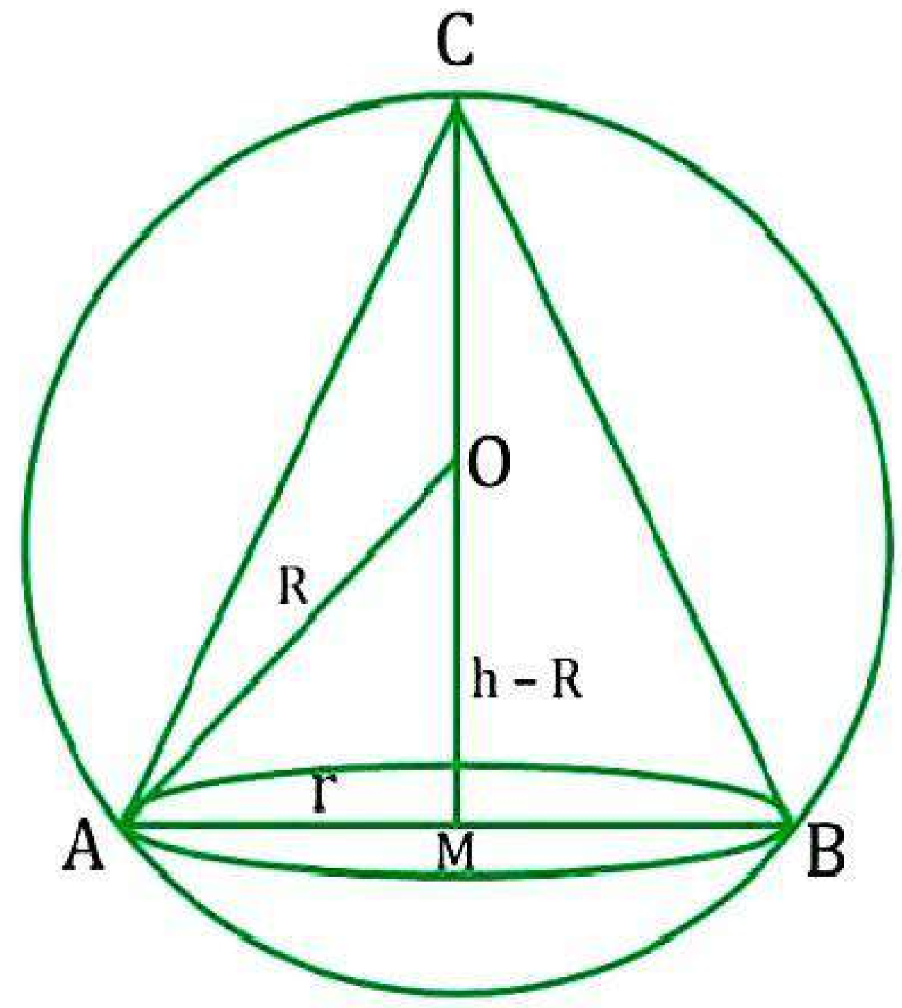
Let r and h be the radius and height of the cone respectively inscribed in a sphere of radius R.

Let V be the volume of the cone

Then $V = \frac{1}{3}\pi r^2 h$

Height of the cone is given by, h = R + AB = $R + \sqrt{R^2 - r^2}$ [ABC is a right triangle]

$$V = \frac{1}{3}\pi r^2\left(R + \sqrt{R^2 - r^2}\right) = \frac{1}{3}\pi r^2 R + \frac{1}{3}\pi r^2\sqrt{R^2 - r^2}$$



$$\frac{dV}{dr} = \frac{2}{3}\pi rR + \frac{2}{3}\pi r\sqrt{R^2 - r^2} + \frac{1}{3}\pi r^2.\frac{(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi rR + \frac{2}{3}\pi r\sqrt{R^2 - r^2} - \frac{1}{3}\pi.\frac{r^3}{2\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi rR + \frac{2\pi r(R^2 - r^2) - \pi r^3}{3\sqrt{R^2 - r^2}} = \frac{2}{3}\pi rR + \frac{2\pi rR^2 - 3\pi r^3}{3\sqrt{R^2 - r^2}}$$

$$\frac{d^2A}{dl^2} = \frac{2\pi R}{3} + \frac{3\sqrt{R^2 - r^2}\,(2\pi R^2 - 9\pi r^2) - (2\pi rR^2 - 3\pi r^3).\frac{(-2r)}{6\sqrt{R^2 - r^2}}}{9(R^2 - r^2)}$$

$$= \frac{2}{3}\pi R + \frac{9(R^2 - r^2)\,(2\pi R^2 - 9\pi r^2) + 2\pi r^2 R^2 + 3\pi r^4}{27(R^2 - r^2)^{\frac{3}{2}}}$$

Now, $\frac{dV}{dr} = 0 \Rightarrow \frac{2}{3}\pi rR = \frac{3\pi r^3 - 2\pi rR^2}{3\sqrt{R^2 - r^2}} \Rightarrow 2R = \frac{3r^2 - 2R^2}{\sqrt{R^2 - r^2}} \Rightarrow 2R\sqrt{R^2 - r^2} = 3r^2 - 2R^2$

$$\Rightarrow 4R^2(R^2 - r^2) = (3r^2 - 2R^2)^2 \quad \Rightarrow 4R^4 - 4R^2r^2 = 9r^2 + 4R^4 - 12r^4R^4$$

$$\Rightarrow 9r^4 = 8R^2r^2 \quad r^2 = \frac{8}{9}R^2$$

When $r^2 = \frac{8}{9}R^2$, then $\frac{d^2A}{dl^2} < 0$

$\therefore$ By second derivative test, the volume of the cone is the maximum when $r^2 = \frac{8}{9}R^2$.

When $r^2 = \frac{8}{9}R^2$, $h = R + \sqrt{R^2 - \frac{8}{9}R^2} = R + \sqrt{\frac{1}{9}R^2} = R + \frac{R}{3} = \frac{4}{3}R$

Therefore $V = \frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(\frac{4}{3}R\right)$

$$= \frac{8}{27}\left(\frac{4}{3}\pi R^3\right) = \frac{8}{27} \times \text{(Volume of the sphere)}$$

Hence, the volume of the largest cone that be inscribed in the sphere is $\frac{8}{27}$ the volume of the sphere.

Question 24:

Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

Answer 24:

Let r and h be the radius and the height (altitude) of the cone respectively.

Then, the volume (V) of the cone is given as:

$V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{\pi r^2}$

The surface area (S) of the cone is given by, $S = \pi r l$ (where l is the slant height)

$$= \pi r\sqrt{r^2 + \frac{9V^2}{\pi^2 r^4}} = \pi r\frac{r\sqrt{\pi^2 r^6 + 9V^2}}{\pi r^2} = \frac{1}{r}\sqrt{\pi^2 r^6 + 9V^2}$$

$$\therefore \frac{dS}{dr} = \frac{r.\frac{6\pi^2 r^5}{2\pi\sqrt{\pi^2 r^6 + 9V^2}} - \sqrt{\pi^2 r^6 + 9V^2}}{r^2} = \frac{3\pi^2 r^6 - \pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}} = \frac{2\pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}}$$

Now, $\frac{dS}{dr} = 0 \Rightarrow 2\pi^2 r^6 = 9V^2$

$r^6 = \frac{9V^2}{2\pi^2}$

Thus, it can be easily verified that when $r^6 = \frac{9V^2}{2\pi^2}, \frac{d^2S}{dr^2} > 0.$

$\therefore$ By second derivative test, the surface area of the cone is the least when $r^6 = \frac{9V^2}{2\pi^2}$.

When $r^6 = \frac{9V^2}{2\pi^2}$, $h = \frac{3V}{\pi r^2} = \frac{3}{\pi r^2}\left(\frac{2\pi^2 r^6}{9}\right)^{\frac{1}{2}} = \frac{3}{\pi r^2}.\frac{\sqrt{2}\,\pi r^2}{3} = \sqrt{2}\,r$

Hence, for a given volume, the right circular cone of the least curved surface has an altitude equal to $\sqrt{2}$ times the radius of the base.

Question 25:

Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $tan^{-1}\sqrt{2}$

Answer 25:

Let θ be the semi-vertical angle of the cone. It is clear that $\theta \in \left[0, \frac{\pi}{2}\right]$

Let r, h, and l be the radius, height, and the slant height of the cone respectively.

The slant height of the cone is given as constant.

Now, $r = l\sin\theta$ and $h = l\cos\theta$

The volume (V) of the cone is given by $V = \frac{1}{3}\pi r^2 h$

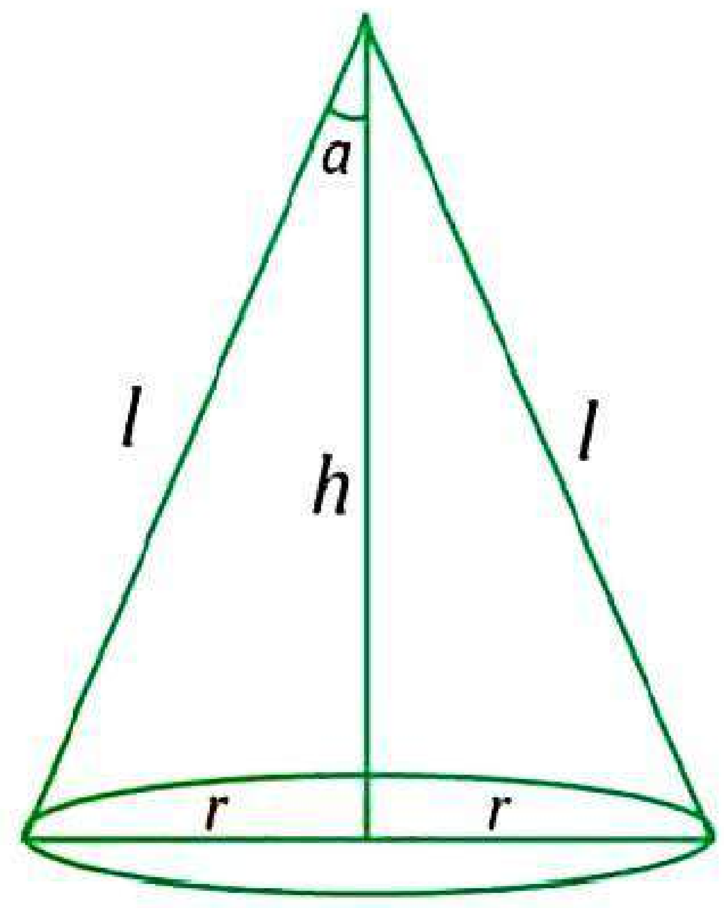
$$= \frac{1}{3}\pi\,(l^2 sin^2\theta)(l\cos\theta) = \frac{1}{3}\pi l^3 sin^2\theta\cos\theta$$

$$\frac{dV}{d\theta} = \frac{\pi l^3}{3}[sin^2\theta(-\sin\theta) + \cos\theta(2\sin\theta\cos\theta]$$

$$= \frac{\pi l^3}{3}[-sin^3\theta + 2\sin\theta\, cos^2\theta\,]$$

$$\frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}[-3sin^2\theta\,\cos\theta + 2cos^3\theta - 4sin^2\theta\cos\theta]$$

$= \frac{\pi l^3}{3}[2cos^2\theta - 7sin^2\theta \ \cos\theta]$



Now, $\frac{dV}{d\theta} = 0$

$sin^3\theta = 2\sin\theta \, cos^2\theta$ so $\quad tan^2\theta = 2 \quad \Rightarrow \tan\theta = \sqrt{2} \quad \Rightarrow \theta = tan^{-1}\sqrt{2}$

Now, when $\theta = tan^{-1}\sqrt{2}$, then $tan^2\theta = 2$ or $sin^2\theta = 2cos^2\theta$. Then we have:

$$\frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}[2cos^3\theta - 14cos^3\theta\,] = -4\pi l^3 cos^3\theta < 0 \; for \;\; \theta \in \left[0, \frac{\pi}{2}\right]$$

By second derivative test, the volume (V) is the maximum when $\theta = tan^2\sqrt{2}$

Hence, for a given slant height, the semi-vertical angle of the cone of the maximum volume is $tan^2\sqrt{2}$

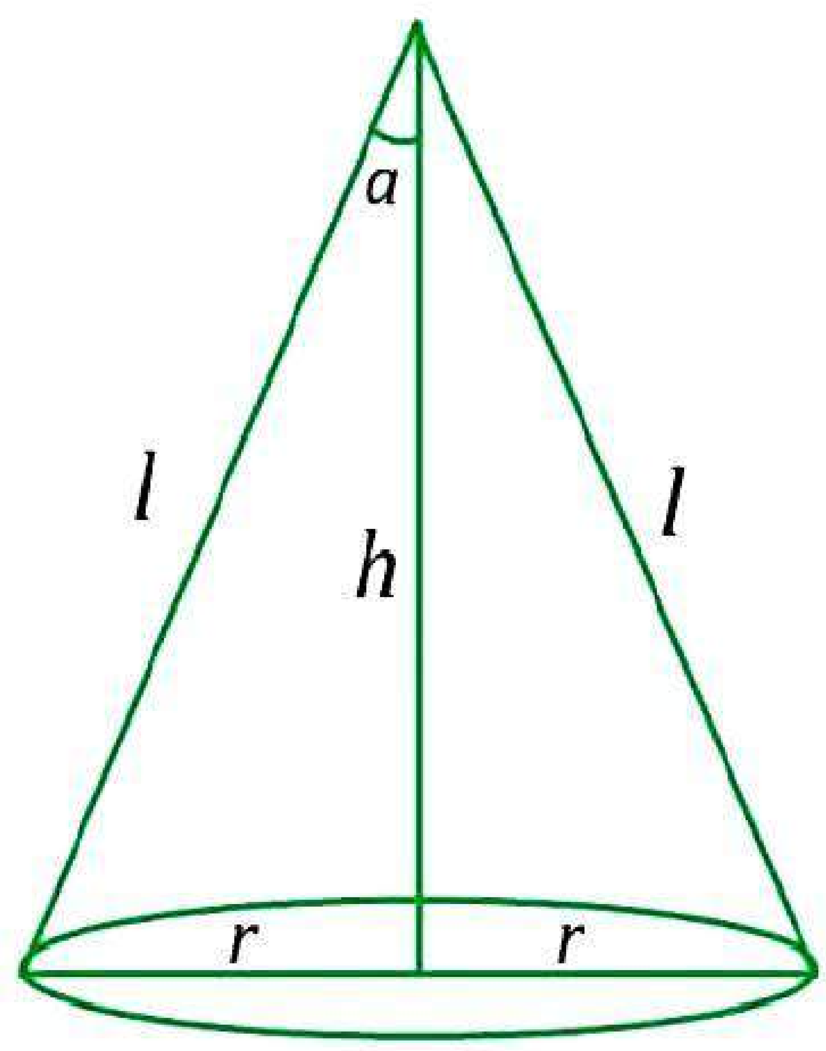
Questions 26:

Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

Answer 26:

Let the height, radius and slant height of the cone be h, r and l respectively, whose semi-vertical angle is a.

Surface area of cone $S = \pi rl + \pi r^2 = \pi r\sqrt{r^2 + h^2} + \pi r^2$



$$\Rightarrow \sqrt{r^2 + h^2} = \frac{S - \pi r^2}{\pi r} \qquad \Rightarrow r^2 + h^2 = \left(\frac{S - \pi r^2}{\pi r}\right)^2$$

$$\Rightarrow h^2 = \left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2 = \frac{S^2 + \pi^2 r^4 - 2S\pi r^2 - \pi^2 r^4}{\pi^2 r^2}$$

$$\Rightarrow h^2 = \frac{S^2 - 2S\pi r^2}{\pi^2 r^2} \qquad \ldots(1)$$

Volume of the cone V = $\frac{1}{3}\pi r^2 h$

$$\Rightarrow V^2 = \frac{1}{9}\pi^2 r^4 h^2$$

$$\Rightarrow V^2 = \frac{1}{9}\pi^2 r^4 \left(\frac{S^2 - 2S\pi r^2}{\pi^2 r^2}\right) = \frac{1}{9}S(Sr^2 - 2\pi r^4)$$

Let, $V^2 = M = \frac{1}{9}S(Sr^2 - 2\pi r^4)$

Therefore, $M'(r) = \frac{1}{9}S(2Sr - 8\pi r^3)$

Now, $M'(r) = 0 \Rightarrow \frac{1}{9}S(2Sr - 8\pi r^3) = 0 \quad \Rightarrow \frac{1}{9}2Sr(S - 4\pi r^2) = 0$

$$\Rightarrow r = 0 \text{ or } r^2 = \frac{S}{4\pi} \quad \Rightarrow r = 0 \text{ or } r = \sqrt{\frac{S}{4\pi}}$$

[$r \neq 0$, as r is the radius of cone, so $r = 0$ is not possible]

Now, $M''(r) = \frac{1}{9}S(2S - 24\pi r^2)$

For $r=\sqrt{\frac{S}{4\pi}}$, $M''\left(\sqrt{\frac{S}{4\pi}}\right)=\frac{1}{9}S\left[2S-24\pi\left(\frac{S}{4\pi}\right)\right]=-\frac{22}{9}S^2<0$

Here, $M''\left(\sqrt{\frac{S}{4\pi}}\right)<0$, therefore $r=\sqrt{\frac{S}{4\pi}}$ is the point of local maxima.

At this maximum point, radius of cone $r=\sqrt{\frac{S}{4\pi}}$ $\quad\Rightarrow r^2=\frac{S}{4\pi}$ $\quad\Rightarrow 4\pi r^2=S$

$\Rightarrow 4\pi r^2=\pi rl+\pi r^2 \Rightarrow 3r=l$ $\qquad\Rightarrow \frac{r}{l}=\frac{1}{3}$

Here, $\quad \sin a=\frac{r}{l}=\frac{1}{3} \qquad \Rightarrow a=\sin^{-1}\left(\frac{1}{3}\right)$

Choose the correct answer in Questions 27 and 29.

Question 27:

The point on the curve $x^2=2y$ which is nearest to the point (0, 5) is

(A) $(2\sqrt{2},4)$ (B) $(2\sqrt{2},0)$ (C) $(0,0)$ (D) $(2,2)$

Answer 27:

The given curve is $x^2=2y$

For each of x, the position of the point will be $\left(x,\frac{x^2}{2}\right)$

The distance d(x) between the point $\left(x,\frac{x^2}{2}\right)$ and (0, 5) is given by,

$$d(x)=\sqrt{(x-0)^2+\left(\frac{x^2}{2}-5\right)^2}=\sqrt{x^2+\frac{x^4}{4}+25-5x^2}=\sqrt{\frac{x^4}{4}-4x^2+25}$$

$$d'(x)=\frac{(x^3-8x)}{2\sqrt{\frac{x^4}{4}-4x^2+25}}=\frac{(x^3-8x)}{\sqrt{x^4-16x^2+100}}$$

Now, $d'(x)=0 \Rightarrow x^3-8x=0 \quad\Rightarrow x(x^2-8)=0 \quad\Rightarrow x=0,\pm2\sqrt{2}$

And $d''(x)=\dfrac{\sqrt{x^4-16x^2+100}(3x^2-8)-2(x^3-8x)-2(x^3-8x)^2}{(x^4-16x^2+100)^{\frac{3}{2}}}$

$$=\frac{(x^4-16x^2+100)(3x^2-8)-2(x^3-8x)(x^3-8x)}{(x^4-16x^2+100)^{\frac{3}{2}}}$$

$$=\frac{(x^4-16x^2+100)(3x^2-8)-2(x^3-8x)^2}{(x^4-16x^2+100)^{\frac{3}{2}}}$$

When, x = 0 then $d''(x)=\frac{36(-8)}{6^3}<0$ and when, $x=\pm2\sqrt{2}$, $d''(x)>0$.

∴By second derivative test, d(x) is the minimum at $x=\pm2\sqrt{2}$

When $x=\pm2\sqrt{2}$, $y=\frac{(2\sqrt{2})^2}{2}=4$

Hence, the point on the curve $x^2=2y$ which is nearest to the point (0, 5) is $(\pm2\sqrt{2},4)$.

The correct answer is (A).

Question 28:

For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is

(A) 0 (B) 1 (C) 3 (D) $\frac{1}{3}$

Answer 28:

Let $y = \frac{1-x+x^2}{1+x+x^2}$

On differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{(1+x+x^2)(-1+2x)-(1-x+x^2)(1+2x)}{(1+x+x^2)^2}$

$= \frac{-1+2x-x+2x^2-x^2+2x^3-1-2x+x+2x^2-x^2-2x^3}{(1+x+x^2)^2}$

$= \frac{-2+2x^2-2x^2+2x^2}{(1+x+x^2)^2} = \frac{2x^2-2}{(1+x+x^2)} = \frac{2(x^2-1)}{(1+x+x^2)^2}$

Put $\frac{dy}{dx} = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

Now, $\frac{d^2y}{dx^2} = \frac{2(1+x+x^2)^2(2x)-(x^2-1)(2)(1+x+x^2)(1+2x)}{(1+x+x^2)^4}$

$= \frac{4(1+x+x^2)(1+x+x^2)x-(x^2-1)(1+2x)}{(1+x+x^2)^4}$

$= \frac{4[x+x^2+x^3-x^2-2x^3+1+2x]}{(1+x+x^2)} = \frac{4(1+3x-x^3)}{(1+x+x^2)^3}$

At x=1, $\left(\frac{d^2y}{dx^2}\right)_{x=1} = \frac{4(1+3(1)-1^3)}{(1+1+1^2)^3} = \frac{4(3)}{3^3} = \frac{4}{9} > 0$

At x=-1, $\left(\frac{d^2y}{dx^2}\right)_{x=1} = \frac{4(1+3(-1)-(-1))^3}{[1+(-1)+(-1)^2]^3} = \frac{4(1-3+1)}{(1-1+1)^3} = 4(-1) = -4 < 0$

∴ By second derivative test, f is minimum at x=1 and the minimum value is given by $y = \frac{1-1+1}{1+1+1} = \frac{1}{3}$

Hence, (d) is the correct option.

Question 29:

The maximum value of $[x(x-1)+1]^{\frac{1}{3}}, 0 \le x \le 1$ is

(A) $\left(\frac{1}{3}\right)^{\frac{1}{3}}$ (B) $\frac{1}{2}$ (C) 1 (D) 0

Answer 29:

$f(x) = [x(x-1)+1]^{\frac{1}{3}}$ ∴ $f(x) = \frac{2x-1}{[x(x-1)+1]^{\frac{1}{3}}}$

Now, $f'(x) = 0 \Rightarrow x = \frac{1}{2}$

Then, we evaluate the value of f at critical point $x = \frac{1}{2}$ and at the end points of the Interval $[0,1]$ $\{i.e.$ at $x = 0$ and $x = 1\}$

$f(0) = [0(0-1)+1]^{\frac{1}{3}} = 1,$ $f(1) = [1(1-1)+1]^{\frac{1}{3}} = 1,$ $f\left(\frac{1}{2}\right) = \left[\frac{1}{2}\left(\frac{1}{2}-1\right)+1\right]^{\frac{1}{3}} = \left(\frac{3}{4}\right)^{\frac{1}{3}}$

Hence, we can conclude that the maximum value of f in the interval [0, 1] is 1.

The correct answer is (C).

Mathematics

(Chapter – 6) (Application of Derivatives) (Miscellaneous Exercise)
(Class – XII)

Question 1:

Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at x = e.

Answer 1:

The given function is $f(x) = \frac{\log x}{x}$

$f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x}{x^2} = \frac{1 - \log x}{x^2}$

Now, $f'(x) = 0$

$1 - \log x = 0 \quad \Rightarrow \quad \log x = 1$

$\log x = \log e \quad \therefore x = e$

Now, $f''(e) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x)(2x)}{x^4}$

$= \frac{x - 2x(1 - \log x)}{x^4} = \frac{-3 + 2\log x}{x^3}$

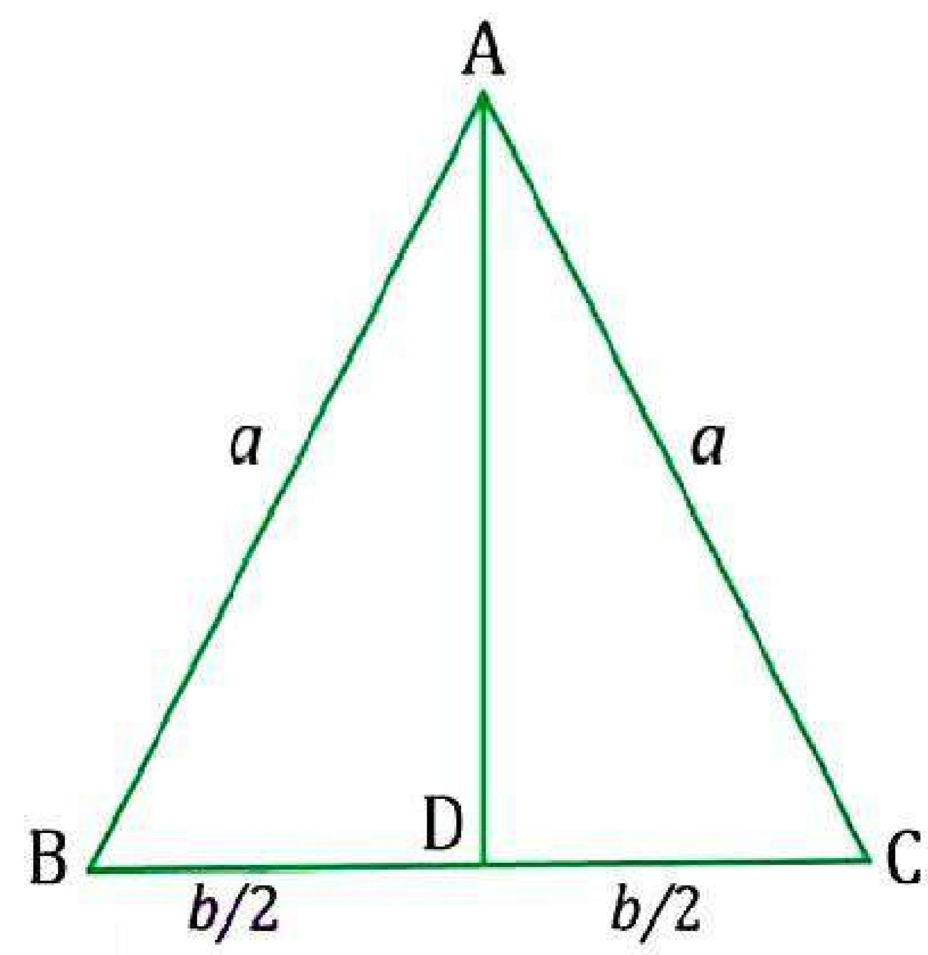
Now, $f''(e) = \frac{-3 + 2\log e}{e^3} = \frac{-3 + 2}{e^3} = \frac{-1}{e^3} < 0$

Therefore, by second derivative test, f is the maximum at $x = e$.

Question 2:

The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second .how fast is the area decreasing when the two equal sides are equal to the base?

Answer 2:



Let ΔABC be isosceles where BC is the base of fixed length b. Let the length of the two equal sides of ΔABC be a draw AD$\perp BC$

Now, in ΔADC, by applying the Pythagoras theorem, we have,

AD = $\sqrt{a^2 - \frac{b^2}{4}}$

Area of triangle (A) = $\frac{1}{2} b \sqrt{a^2 - \frac{b^2}{4}}$

The rate of change of the area with respect to time (t) is given by,

$$\frac{dA}{dt} = \frac{1}{2} b . \frac{2a}{2\sqrt{a^2 - \frac{b^2}{4}}} \frac{da}{dt} = \frac{ab}{\sqrt{4a^2 - b^2}} \frac{da}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of 3 cm per second.

$\frac{da}{dt} = -3\ cm/sec$

$$\Rightarrow \frac{dA}{dt} = \frac{-3ab}{\sqrt{4a^2 - b^2}}$$

Then, when $a = b$ we have:

$$\frac{dA}{dt} = \frac{-3ab}{\sqrt{4a^2 - b^2}} = \frac{-3b^2}{\sqrt{3b^2}} = -\sqrt{3}\, b$$

Hence, if the two equal sides are equal to the base, then the area of the triangle is decreasing at the rate of $\sqrt{3}\, b$ cm/sec^2.

Question 3:

Find the intervals in which the function f given by $f(x)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$ is (i) Increasing (ii) Decreasing.

Answer 3:

$$f(x)=\frac{4\sin x-2x-x\cos x}{2+\cos x}$$

$$\Rightarrow f'(x)=\frac{(2+\cos x)(4\cos x-2-\cos x+x\sin x)-(4\sin x-2x-x\cos x)(-\sin x)}{(2+\cos x)^2}$$

$$\Rightarrow f'(x)=\frac{(2+\cos x)(3\cos x-2+x\sin x)+\sin x(4\sin x-2x-x\cos x)}{(2+\cos x)^2}$$

$$\Rightarrow f'(x)=\frac{6\cos x-4+2x\sin x+3cos^2x-2\cos x+x\sin x cos\, x+4sin^2x-2x\sin x-x\, sinx\cos x}{(2+\cos x)^2}$$

$$\Rightarrow f'(x)=\frac{4\cos x-4+3cos^2x+4sin^2x}{(2+\cos x)^2}\Rightarrow f'(x)=\frac{4\cos x-4+3cos^2x+4-4cos^2x}{(2+\cos x)^2}$$

$$\Rightarrow f'(x)=\frac{4\cos x-cos^2x}{(2+\cos x)^2}=\frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

Now $f'(x)=0\ \Rightarrow\ \cos x=0$ or $\cos x=4$

But $\cos x\neq 4$

$\cos x=0\ \ i.e.\ \ x=\frac{\pi}{2},\frac{3\pi}{2}$

Now, $x=\frac{\pi}{2}$ and $x=\frac{3\pi}{2}$ divides $(0,2\pi)$ into three disjoint intervals, i.e.

$\left(0,\frac{\pi}{2}\right),\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ and $\left(\frac{3\pi}{2},2\pi\right)$

In intervals $\left(0,\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, $\quad f'(x)>0$.

Thus, $f(x)$ is increasing for $0<x<\frac{\pi}{2}$ and $\frac{3\pi}{2}<x<2\pi$.

In the interval $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, $f'(x)<0$.

Thus, $f(x)$ is decreasing for $\frac{\pi}{2}<x<\frac{3\pi}{2}$.

Question 4:

Find the intervals in which the function f given by $f(x)=x^3+\frac{1}{x^3},\ x\neq 0$ is (i) increasing (ii) decreasing.

Answer 4:

$$f(x)=x^3+\frac{1}{x^3}\quad\Rightarrow f'(x)=3x^2-\frac{3}{x^4}=\frac{3x^6-3}{x^4}$$

Then, $f'(x)=0\ \ i.e. 3x^6-3=0$

$\Rightarrow x^6=1,\ \Rightarrow x=\pm 1$

Now, the points $x=1$ and $x=-1$ divide the real line into three disjoint intervals i.e., $(-\infty,-1),(-1,1)$ and $(1,\infty)$

In intervals $(-\infty,-1)$ and $(1,\infty)$ i.e., when $x<-1$ and $x>1$, $f'(x)>0$.

Thus, when $x<-1$ and $x>1$, f is increasing.

In interval $(-1, 1)$ i.e., when $-1 < x < 1$, $f'(x) < 0$

Thus, when $-1<x<1$, f is decreasing.

Question 5:

Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis.

Answer 5:

The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let the major axis be along the x −axis.

Let ABC be the triangle inscribed in the ellipse where vertex C is at (a, 0).

Since the ellipse is symmetrical with respect to the x−axis and y −axis, we can assume the coordinates of A to be $(-x_1, y_1)$ and the coordinates of B to be $(-x_1, -y_1)$.

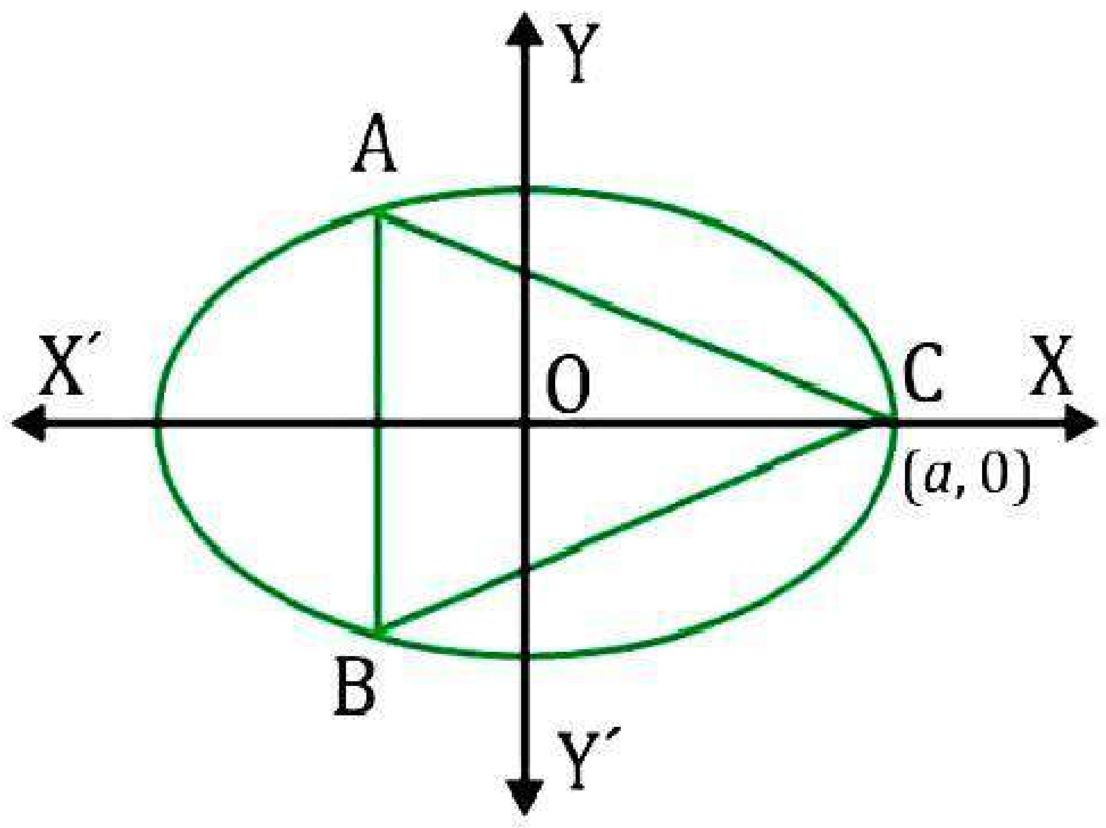
Now we have $y_1 = \pm\frac{b}{a}\sqrt{a^2-x_1^2}$

Coordinate of A are $\left(x_1, \frac{b}{a}\sqrt{a^2-x_1^2}\right)$ and the coordinate of B are $\left(x_1, -\frac{b}{a}\sqrt{a^2-x_1^2}\right)$.

As the point (x_1, y_1) lies on the ellipse, the area of triangle ABC is given by,

$$A=\frac{1}{2}\left|a\left(\frac{2b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)\right|$$

$$\Rightarrow A=b\sqrt{a^2-x_1^2}+x_1\frac{b}{a}\sqrt{a^2-x_1^2} \qquad \dots(1)$$



$$\frac{dA}{dx_1}=\frac{-2x_1b}{2\sqrt{a^2-x_1^2}}+\frac{b}{a}\sqrt{a^2-x_1^2}-\frac{2bx_1^2}{2a\sqrt{a^2-x_1^2}}$$

$$\Rightarrow\frac{dA}{dx_1}=\frac{b}{a\sqrt{a^2-x_1^2}}\left[-x_1a+(a^2-x_1^2)-x_1^2\right]$$

$$\Rightarrow\frac{dA}{dx_1}=\frac{b(-2x_1^2-x_1a+a^2}{a\sqrt{a^2-x_1^2}}$$

Now $\frac{dA}{dx_1}=0 \quad \Rightarrow -2x_1^2-x_1a+a^2=0$

$$\Rightarrow x_1=\frac{a\pm\sqrt{a^2-4(-2)(a^2)}}{2(-2)}=\frac{a\pm\sqrt{9a^2}}{-4}=\frac{a\pm 3a}{-4}$$

$$\Rightarrow x_1=-a, \qquad \frac{a}{2}.$$

But, x_1 cannot be equal to $-a$.

$$x_1=\frac{a}{2}=\frac{b}{a}\left\{\sqrt{a^2-\frac{b^2}{4}}\right\}\Rightarrow\frac{ba}{2a}\sqrt{3}\Rightarrow\frac{\sqrt{3}\,b}{2}$$

$$\text{Now, }\frac{d^2A}{dx_1^2}=\frac{b}{a}\left\{\frac{\sqrt{a^2-x_1^2}(-4x_1-a)-(-2x_1^2-x_1a+a^2)\frac{(-2x_1)}{2\sqrt{a^2-x_1^2}}}{a^2-x_1^2}\right\}$$

$$=\frac{b}{a}\left\{\frac{(a^2-x_1^2)\,(-4x_1-a)+x_1\,(-2x_1^2-x_1a+a^2)}{(a^2-x_1^2)^{\frac{3}{2}}}\right\}=\frac{b}{a}\left\{\frac{2x^3-3a^2x-a^3}{(a^2-x_1^2)^{\frac{3}{2}}}\right\}$$

Also, when $x_1 = \frac{a}{2}$, then

$$\frac{d^2A}{dx_1^2} = \frac{b}{a}\left\{\frac{2\frac{a^3}{8} - 3\frac{a^3}{2} - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\} = \frac{b}{a}\left\{\frac{\frac{a^3}{4} - \frac{3}{2}a^3 - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\} = -\frac{b}{a}\left\{\frac{\frac{9}{4}a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\} < 0$$

Thus, the area is the maximum when $x_1 = \frac{a}{2}$

Maximum area of the triangle is given by,

$$A = b\sqrt{a^2 - \frac{b^2}{4}} + \left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2 - \frac{b^2}{4}} = ab\frac{\sqrt{3}}{2} + \left(\frac{a}{2}\right)\frac{b}{a} \times \frac{a\sqrt{3}}{2} = \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}ab$$

Question 6:

A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m³. If building of tank costs ₹70 per square meters for the base and ₹45 per square metre for sides. What is the cost of least expensive tank?

Answer 6:

Let l, b, and h represent the length, breadth, and height of the tank respectively.

Then, we have height (h) = 2 m

Volume of the tank = 8 m³

Volume of the tank = $l \times b \times h$

$\therefore 8 = l \times b \times 2$

$lb = 4 \quad \Rightarrow b = \frac{4}{l}$

Now, area of the base = lb = 4

Area of the 4 walls (A) = $2h(l + b)$

$A = 4\left(l + \frac{4}{l}\right) \quad \Rightarrow \frac{dA}{dl} = 4\left(1 - \frac{4}{l^2}\right)$

Now, $\frac{dA}{dl} = 0$

$1 - \frac{4}{l^2} = 0$

$\Rightarrow l^2 = 4 \qquad \Rightarrow l = \pm 2$

However, the length cannot be negative. Therefore, we have l = 2

$b = \frac{4}{l} = \frac{4}{2} = 2$

Now $\frac{d^2A}{dl^2} = \frac{32}{l^3}$

When $l = 2$, $\frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0$

Thus, by second derivative test, the area is the minimum when l = 2.

We have $l = b = h = 2$.

$\therefore$ Cost of building the base = ₹ 70 × (lb) = ₹ 70 (4) = ₹ 280

Cost of building the walls = ₹ 2h (l + b) × 45 = ₹ 90 (2) (2 + 2) = ₹ 8 (90) = ₹ 720

Required total cost = ₹ (280 + 720) = ₹ 1000

Hence, the total cost of the tank will be ₹ 1000.

Question 7:

The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

Answer 7:

Let r be the radius of the circle and a be the side of the square.

Then, we have:

$2\pi r + 4a = k \ (\text{where } k \text{ is constant}) \qquad \Rightarrow a = \dfrac{k - 2\pi r}{4}$

The sum of the areas of the circle and the square (A) is given by,

$$A = \pi r^2 + a^2 = \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$

$$\frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(-2\pi)}{16} = 2\pi r - \frac{\pi(k - 2\pi r)}{4}$$

Now $\frac{dA}{dr} = 0$.

$$\Rightarrow 2\pi r = \frac{\pi(k - 2\pi r)}{4} \qquad \Rightarrow 8r = k - 2\,\pi r \qquad \Rightarrow (8 + 2\pi)r = k \qquad \Rightarrow r = \frac{k}{8 + 2\pi} = \frac{k}{2(4 + \pi)}$$

Now, $\frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$.

$\therefore$ When $r = \frac{k}{2(4+\pi)}, \qquad \frac{d^2A}{dr^2} > 0$.

The sum of the area is least when $r = \frac{k}{2(4+\pi)}$

$$a = \frac{k - 2\pi\left[\frac{k}{2(4 + \pi)}\right]}{4} = \frac{k(4 + \pi) - \pi k}{4(4 + \pi)} = \frac{4k}{4(4 + \pi)} = 2r$$

Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

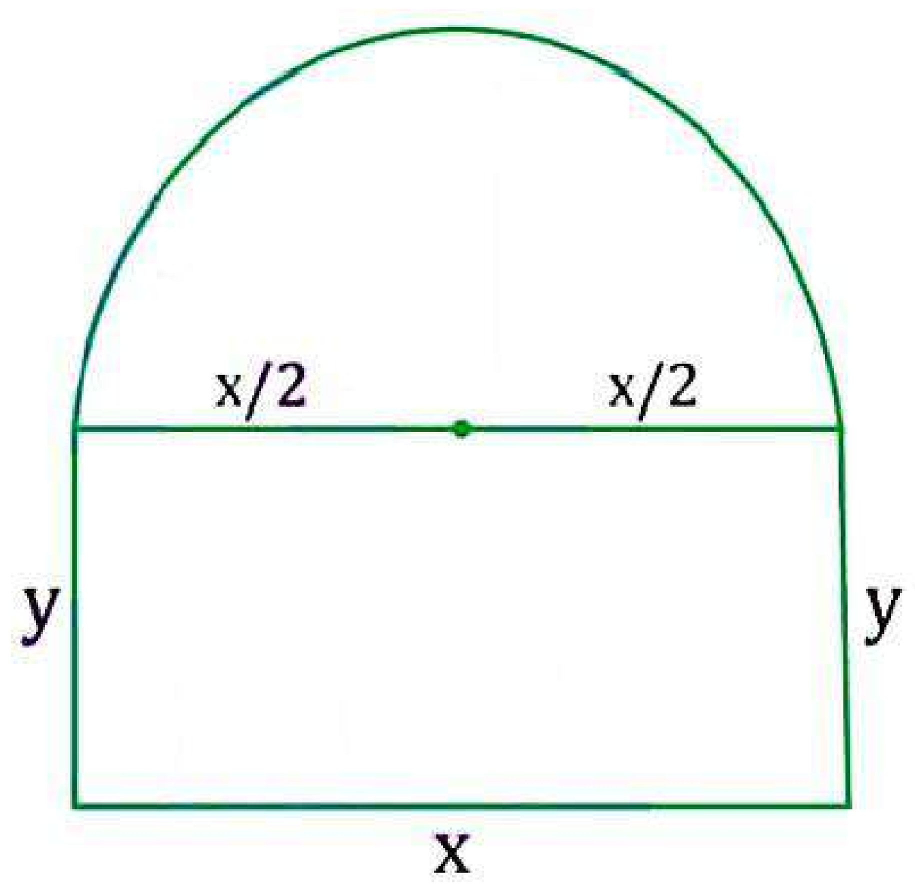
Question 8:

A window is in the form of rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening

Answer 8:

Let x and y be the length and breadth of the rectangular window.

Radius of the semicircular opening = $\frac{x}{2}$



It is given that the perimeter of the window is 10 m.

$$\Rightarrow x + 2y + \frac{\pi x}{2} = 10 \qquad \Rightarrow x\left(1 + \frac{\pi}{2}\right) + 2y = 10$$

$$\Rightarrow 2y = 10 - x\left(1 + \frac{\pi}{2}\right) \qquad \Rightarrow y = 5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)$$

Area of the window (A) is given by,

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$

$$\Rightarrow A = x\left[5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)\right] + \frac{\pi x^2}{8} = 5x - x^2\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi x^2}{8}$$

$$\Rightarrow \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi x}{4} = 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x$$

$$\Rightarrow \frac{d^2A}{dx^2} = -\left(\frac{1}{2}+\frac{\pi}{4}\right)+\frac{\pi}{4} = -1-\frac{\pi}{4}$$

Now, $\frac{dA}{dx} = 0$

$$\Rightarrow 5 - x\left(1+\frac{\pi}{2}\right)+\frac{\pi}{4}x = 0 \qquad \Rightarrow x\left(1+\frac{\pi}{2}\right) = 5$$

$$\Rightarrow x = \frac{5}{\left(1+\frac{\pi}{2}\right)} = \frac{20}{\pi+4}$$

Thus, when $x = \frac{20}{\pi+4}$ $\;$ *then* $\frac{d^2A}{dx^2} < 0.$

Therefore, by second derivative test, the area is the maximum when length $x = \frac{20}{\pi+4}m$. Now,

$$y = 5 - \frac{20}{\pi+4}\left(\frac{2+\pi}{4}\right) = 5 - \frac{5(2+\pi)}{\pi+4} = \frac{10}{\pi+4}m$$

Hence, the required dimensions of the window to admit maximum light is given by

Length $x = \frac{20}{\pi+4}m$ and breath $= \frac{10}{\pi+4}m.$

Question 9:

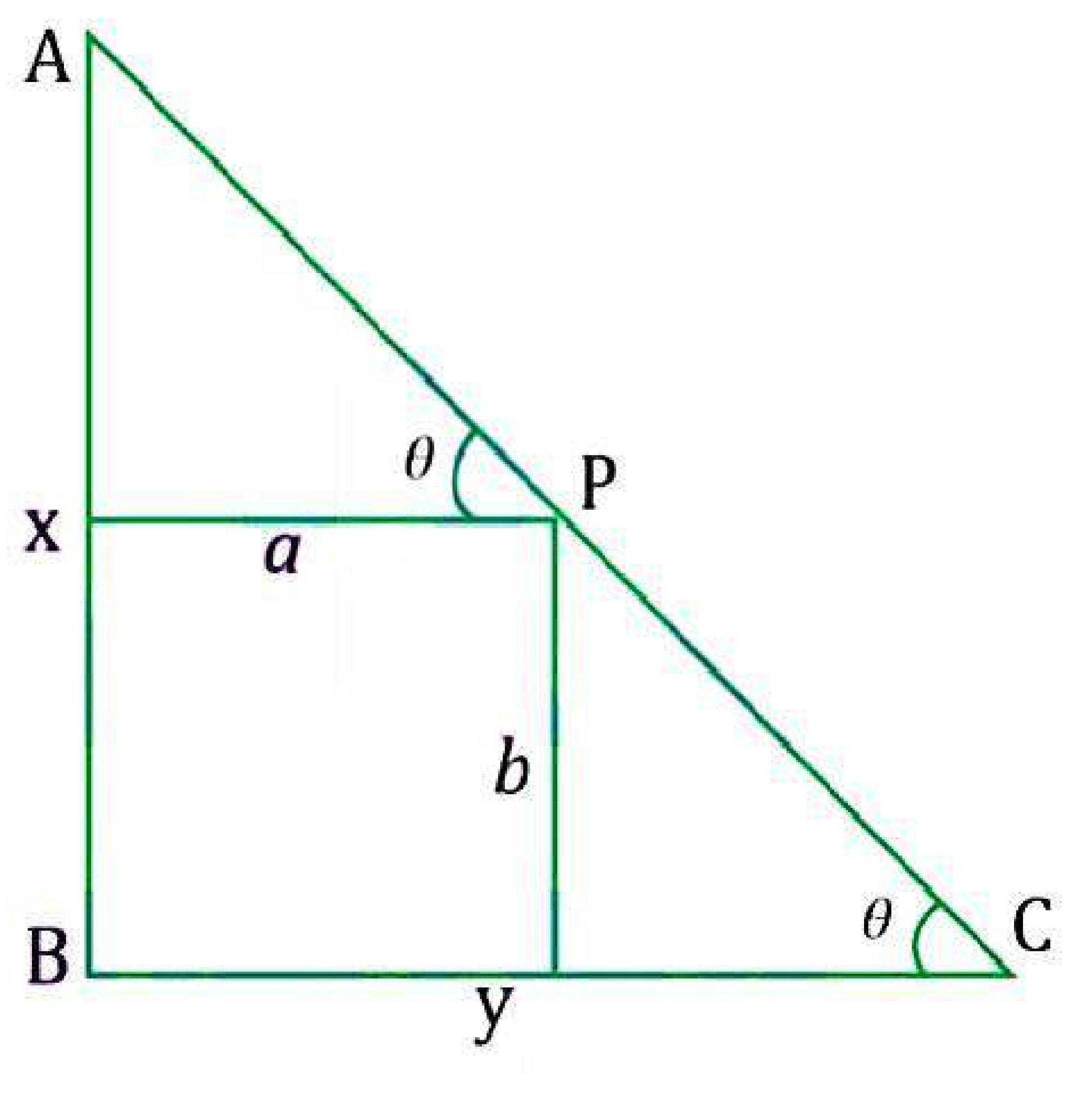
A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Answer 9:

Let ABC be right-angled at B. Let AB = x and BC = y.

Let P be a point on the hypotenuse of the triangle such that P is at a distance of a and b from the sides AB and BC respectively. Let $\angle c = \theta$

We have, AC = $\sqrt{x^2+y^2}$



Now, $PC = b\ cosec\ \theta$ and $AP = a\sec\theta$

$\Rightarrow$ AC = AP + PC $\Rightarrow AC = a\sec\theta + b\ cosec\ \theta$ ... (1)

$$\frac{d(AC)}{d\theta} = -b\ cosec\ \theta\cot\theta + a\sec\theta\tan\theta$$

$$\therefore \frac{d(AC)}{d\theta} = 0 \quad \Rightarrow a\sec\theta\tan\theta = b\ cosec\ \theta\cot\theta$$

$$\Rightarrow \frac{a}{\cos\theta}.\frac{\sin\theta}{\cos\theta} = \frac{b}{\sin\theta}.\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow a\ sin^3\theta = b\ cos^3\theta \Rightarrow \tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$\therefore \sin\theta = \frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}} \quad and \quad \cos\theta = \frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}}+b^{\frac{2}{3}}}} \qquad ...(2)$$

It can be clearly shown that $\frac{d^2(AC)}{d\theta^2} > 0$ when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$

Therefore, by second derivative test, the length of the hypotenuse is minimum when $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Now, if $\tan\theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, we have:

$$AC = \frac{b\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{b^{\frac{2}{3}}} + \frac{b\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{a^{\frac{2}{3}}} \qquad \text{[Using (1) and (2)]}$$

$$= \sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right) = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{2}{3}}$$

Hence, the minimum length of the hypotenuses is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{2}{3}}$.

Question 10:

Find the points at which the function f given by $f(x) = (x-2)^4(x+1)^3$ has

(i) local maxima (ii) local minima (iii) point of inflexion.

Answer 10:

The given function is $f(x) = (x-2)^4(x+1)^3$.

$$\therefore f'(x) = 4(x-2)^3(x+1)^3 + 3(x+1)^2(x-2)^4$$

$$= (x-2)^3(x+1)^2\left[4(x+1)+3(x-2)\right]$$

$$= (x-2)^3(x+1)^2(7x-2)$$

Now, $f'(x)=0 \Rightarrow x=-1$ and $x=\frac{2}{7}$ or $x=2$

Now, for values of x close to $\frac{2}{7}$ and to the left of $\frac{2}{7}$, $f'(x)>0$. Also, for values of x close to $\frac{2}{7}$ and to the right of $\frac{2}{7}$, $f'(x)<0$.

Thus, $x=\frac{2}{7}$ is the point of local maxima.

Now, for values of x close to 2 and to the left of 2, $f'(x)<0$. Also, for values of x close to 2 and to the right of 2, $f'(x)>0$.

Thus, $x = 2$ is the point of local minima.

Now, as the value of x varies through −1, f ' (x) does not changes its sign.

Thus, x = −1 is the point of inflexion.

Question 11:

Find the absolute maximum and minimum values of the function f given by

$f(x) = cos^2 x + \sin x,\ \ x \in [0, \pi]$

Answer 11:

$f(x) = cos^2 x + \sin x$

$\Rightarrow f'(x) = 2\cos x(-sinx) + \cos x$

$\Rightarrow f'(x) = -2\sin x \cos x + \cos x$

Now, $f'(x) = 0 \quad \Rightarrow 2\sin x \cos x = \cos x \quad \Rightarrow \cos x(2\sin x - 1) = 0$

$\Rightarrow \sin x = \frac{1}{2}$ or $\cos x = 0$

$\Rightarrow x = \frac{\pi}{6}$ or $\frac{\pi}{2}$ as $x \in [0, \pi]$

Now, evaluating the value of f at critical points $x = \frac{\pi}{2}$ *and* $x = \frac{\pi}{6}$ and at the end points of the interval $[0, \pi]$ (i.e., at $x = 0$ and $x = \pi$), we have:

$$f\left(\frac{\pi}{6}\right) = cos^2\frac{\pi}{6} + \sin\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4}$$

$f(0) = cos^2 0 + \sin 0 = 1 + 0 = 1$

$f(\pi) = cos^2 \pi + \sin \pi = (-1)^2 + 0 = 1$

$$f\left(\frac{\pi}{2}\right) = cos^2\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$$

Hence, the absolute maximum value of f is $\frac{5}{4}$ occurring at $x = \frac{\pi}{6}$ and the absolute minimum value of f is 1 occurring at $x = 0, \frac{\pi}{2}$ and π.

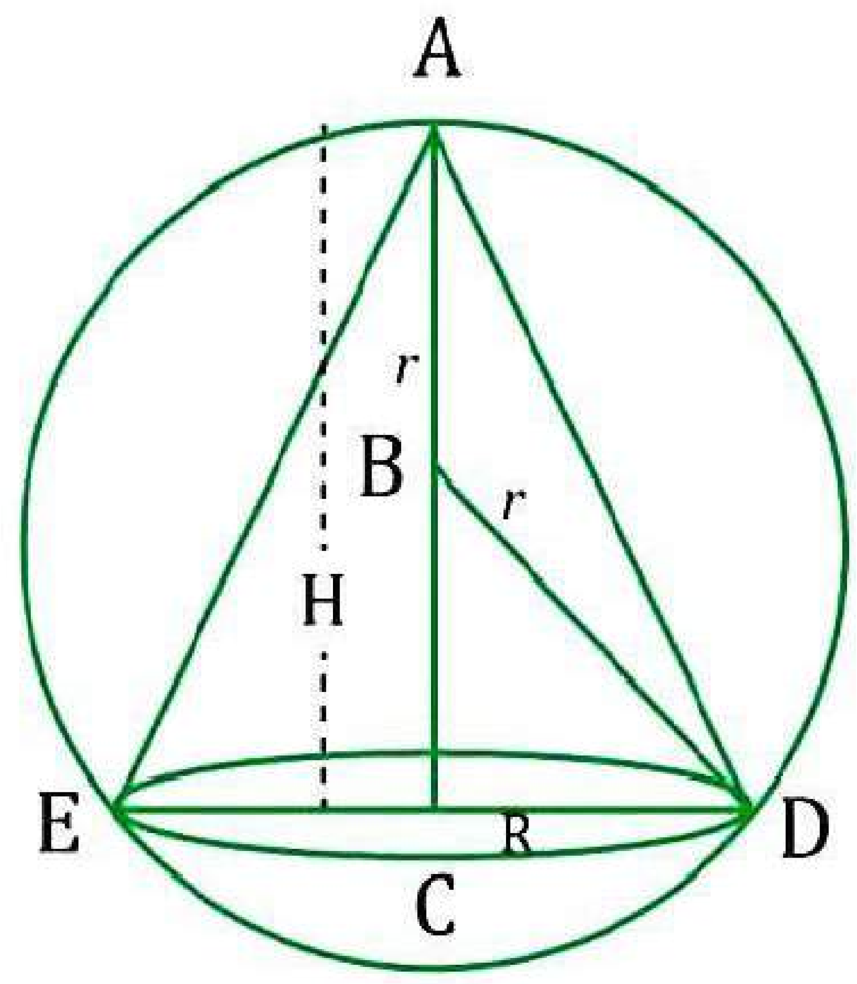
Question 12:

Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Answer 12:

A sphere of fixed radius (r) is given. Let R and H be the radius and the height of the cone respectively.
The volume (V) of the cone is given by,

$$V = \frac{1}{3}\pi R^2 H$$



Now, from the right triangle BCD, we have: $BC = \sqrt{r^2 - R^2}$

$$H = r + \sqrt{r^2 - R^2}$$

$$V = \frac{1}{3}\pi R^2\left(r + \sqrt{r^2 - R^2}\right) = \frac{1}{3}\pi R^2 r + \frac{1}{3}\pi R^2\sqrt{r^2 - R^2}$$

$$\frac{dV}{dR} = \frac{2}{3}\pi R r + \frac{2}{3}\pi R\sqrt{r^2 - R^2} + \frac{R^2}{3}.\frac{(-2R)}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi R r + \frac{2}{3}\pi R\sqrt{r^2 - R^2} - \frac{R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi R r + \frac{2\pi R(r^2 - R^2) - \pi R^3}{3\sqrt{r^2 - R^2}} = \frac{2}{3}\pi R r + \frac{2\pi R r^2 - 3\pi R^3}{3\sqrt{r^2 - R^2}}$$

Now, $\frac{dV}{dR} = 0$

$$\Rightarrow \frac{2}{3}\pi R r = \frac{3\pi R^3 - 2\pi R r^2}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow 4r^2(r^2 - R^2) = (3R^2 - 2r^2)^2 \quad \Rightarrow 4r^4 - 4r^4R^2 = 3R^4 + 4r^4 - 12R^2r^2$$

$$\Rightarrow 9R^4 - 8R^2r^2 = 0 \quad \Rightarrow R^2 = \frac{8r^2}{9}$$

Now, $$\frac{d^2V}{dR^2} = \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}\ (2\pi r^2 - 9\pi R^2) + (2\pi R r^2 - 3\pi R^3)(-6R)\frac{1}{2\sqrt{r^2 - R^2}}}{9(r^2 - R^2)}$$

$$= \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}\ (2\pi r^2 - 9\pi R^2) + (2\pi R r^2 - 3\pi R^3)(3R)\frac{1}{2\sqrt{r^2 - R^2}}}{9(r^2 - R^2)}$$

Now, when $R^2 = \frac{8r^2}{9}$, it can be shown that $\frac{d^2V}{dR^2} < 0$.

The volume is the maximum when $R^2 = \frac{8r^2}{9}$

When $R^2 = \frac{8r^2}{9}$, height of the cone $= r + \sqrt{r^2 - \frac{8r^2}{9}} \quad = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}$

Hence, it can be seen that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Question 13:

Let f be a function defined on $[a, b]$ such that $f'(x) > 0$, for all $x \in (a, b)$. Then prove that f is an increasing function on (a, b).

Answer 13:

We have to prove that function is always increasing, *i.e.*

$f(x_2) > f(x_1)$ for all $x_2 > x_1$. [Where $x_1, x_2 \in [a, b]$]

Let x_1 and x_2 be two numbers in the interval $[a, b]$, *i.e.* $x_1, x_2 \in [a, b]$ and $x_2 > x_1$.

Consider the interval $[x_1, x_2]$.

Function f is continuous as well as differential in $[x_1, x_2]$ as it is continuous and differential in $[a, b]$.

Using Mean value theorem, there exists $c \in [x_1, x_2]$, such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Given that $f'(x) > 0$ for all $x \in [a, b]$.

Therefore, $f'(c) > 0$ for all $x \in [x_1, x_2]$.

$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \qquad \Rightarrow f(x_2) - f(x_1) > 0 \qquad \Rightarrow f(x_2) > f(x_1)$

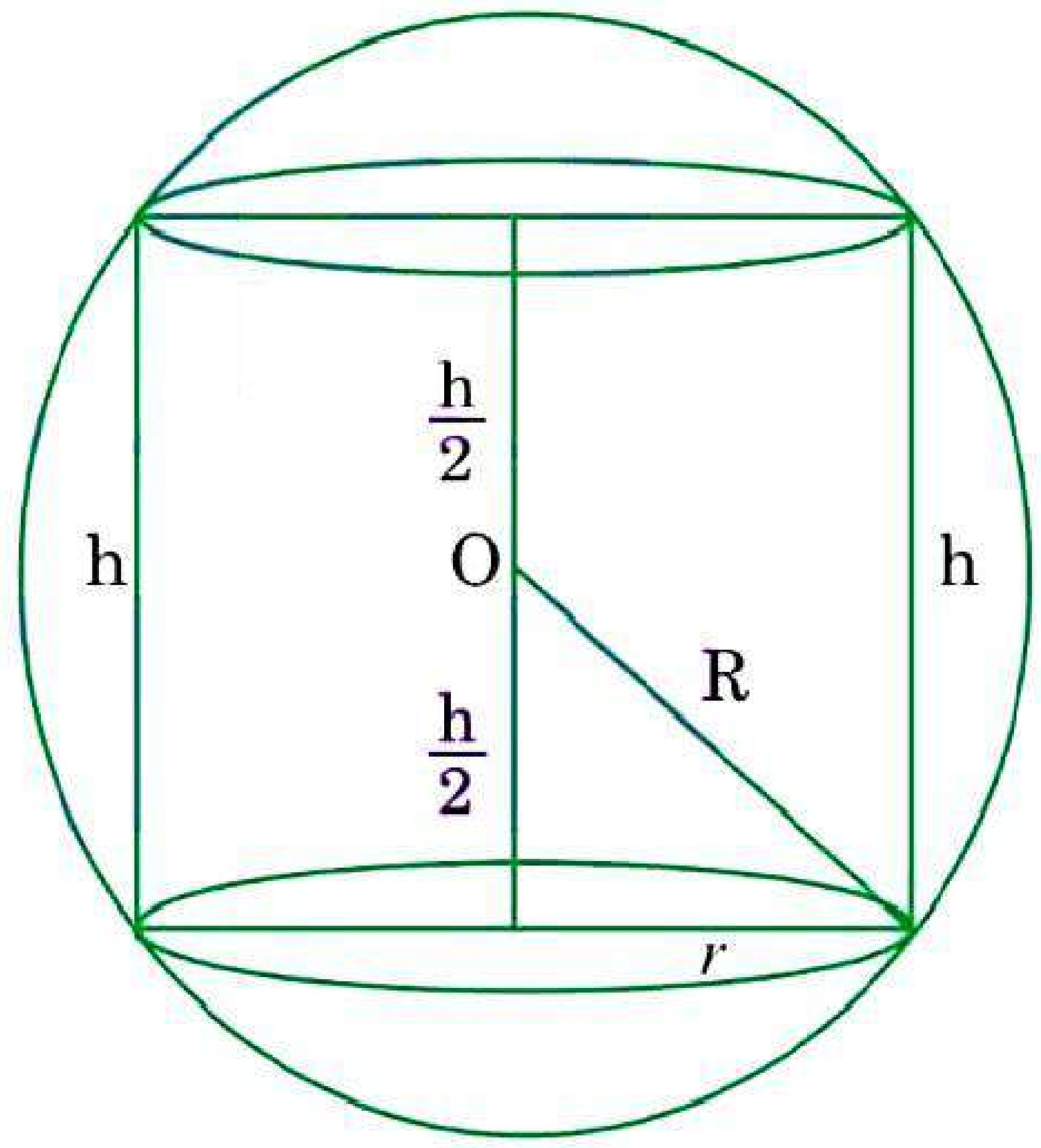
Now, for the two point $x_1, x_2 \in [a, b]$, where $x_2 > x_1$, we have $f(x_2) > f(x_1)$.

Hence, the functions f is increasing in the interval $[a, b]$.

Question 14:

Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

Answer 14:



A sphere of fixed radius (R) is given.

Let r and h be the radius and the height of the cylinder respectively.

From the figure, we have $h = 2\sqrt{R^2 - r^2}$

The volume (V) of the cylinder is given by,

$V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$

$\frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + \frac{2\pi r^2(-2r)}{2\sqrt{R^2 - r^2}}$

$= 4\pi r\sqrt{R^2 - r^2} - \frac{2\pi r^3}{2\sqrt{R^2 - r^2}}$

$$= \frac{4\pi r(R^2 - r^2) - 2\pi r^3}{\sqrt{R^2 - r^2}} = \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

Now, $\frac{dV}{dr} = 0 \quad \Rightarrow 4\pi r R^2 - 6\pi r^3 = 0$

$$r^2 = \frac{2R^2}{3}$$

$$\text{Now,} \frac{d^2V}{dR^2} = \frac{\sqrt{R^2 - r^2}(4\pi R^2 - 18\pi r^2) - (4\pi R^2 - 6\pi r^3)\frac{(-2r)}{2\sqrt{R^2 - r^2}}}{R^2 - r^2}$$

$$= \frac{(R^2 - r^2)(4\pi R^2 - 18\pi r^2) + r(4\pi R^2 - 6\pi r^3)}{(R^2 - r^2)^{\frac{3}{2}}}$$

$$= \frac{4\pi R^2 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{(R^2 - r^2)^{\frac{3}{2}}}$$

Now, it can be observed that at $r^2 = \frac{2R^2}{3}$, $\frac{d^2V}{dR^2} < 0$.

$\therefore$ The volume is the maximum when $r^2 = \frac{2R^2}{3}$

When $r^2 = \frac{2R^2}{3}$, the height of the cylinder $h = 2\sqrt{R^2 - \frac{2R^2}{3}} = \frac{2R}{\sqrt{3}}$

Hence, the volume of the cylinder is the maximum when the height of the cylinder is $\frac{2R}{\sqrt{3}}$

Question 15:

Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 tan^2 \alpha$.

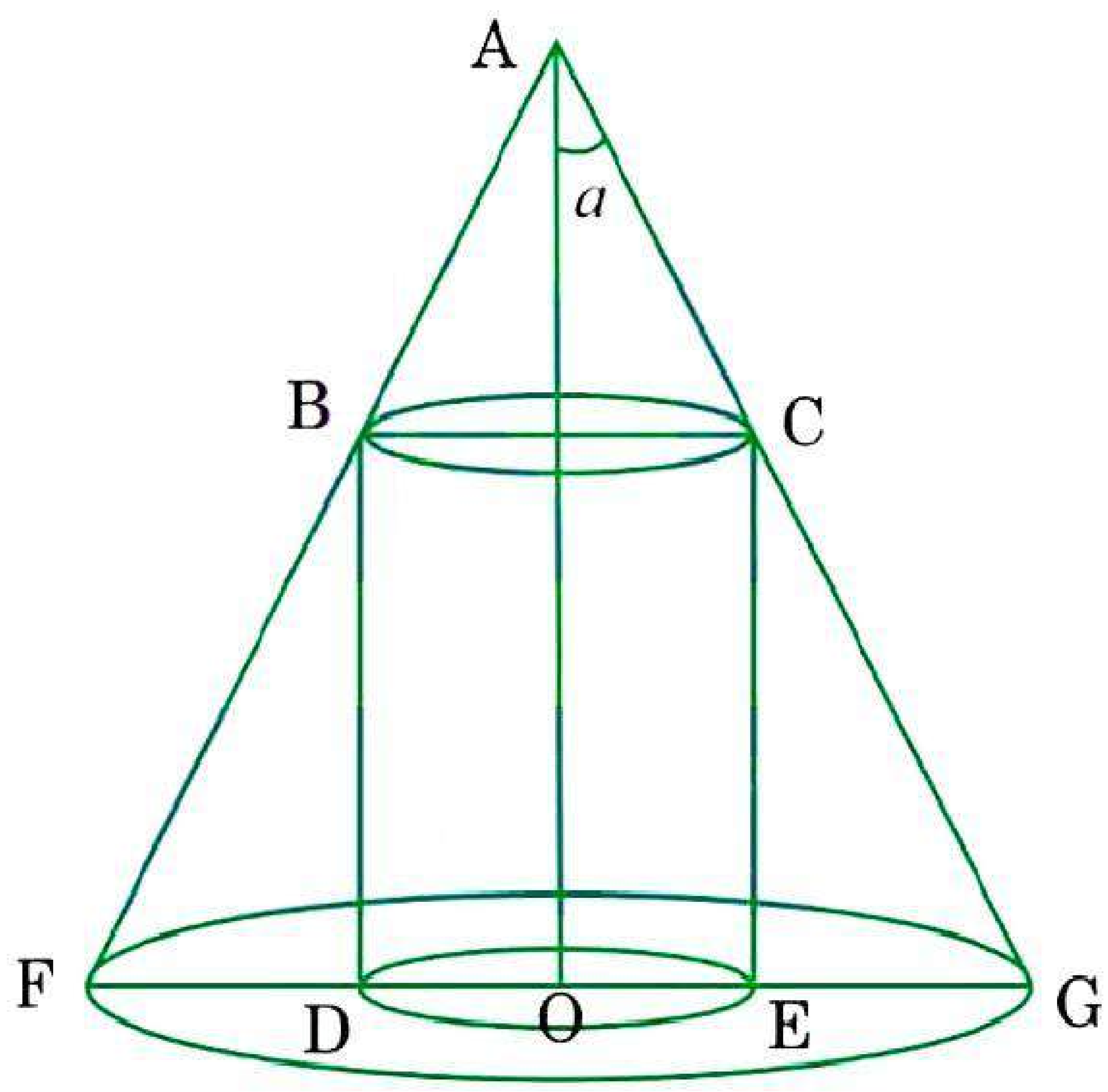
Answer 15:

The given right circular cone of fixed height (h) and semi-vertical angle (α) is given. Here, a cylinder of radius R and height H is inscribed in the cone.

$\therefore$ GAO = α, OG = r, OA = h, OE = R, and CE = H.

We have,

$r = h \tan \alpha$



Now, since ΔAOG is similar to ΔCEG, we have:

$\frac{AO}{OG} = \frac{CE}{EG} \quad \frac{h}{r} = \frac{H}{r-R} \qquad$ [EG = OG – OE]

$$\Rightarrow H = \frac{h}{r}(r - R) = \frac{h}{h\tan\alpha}(h\tan\alpha - R) = \frac{1}{\tan\alpha}(h\tan\alpha - R)$$

Now the volume (V) of the cylinder is given by,

$$V = \pi R^2 H = \frac{\pi R^2}{\tan\alpha}(h\tan\alpha - R) = \pi R^2 h - \frac{\pi R^3}{\tan\alpha}$$

$\Rightarrow \frac{dV}{dR} = 2\pi RH - \frac{3\pi R^2}{\tan\alpha}$

Now, $\frac{dV}{dR} = 0 \quad \Rightarrow 2\pi RH = \frac{3\pi R^2}{\tan\alpha}$

$\Rightarrow 2\text{h} \tan\alpha = 3R \qquad \Rightarrow R = \frac{2h}{3}\tan\alpha$

Now, $\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi R}{\tan\alpha}$

And for $R = \frac{2h}{3}\tan\alpha$, we have: $\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi}{\tan\alpha}\left(\frac{2h}{3}\tan\alpha\right) = 2\pi h - 4\pi h < 0$

∴By second derivative test, the volume of the cylinder is the greatest when $R = \frac{2h}{3}\tan\alpha$,

When $R = \frac{2h}{3}\tan\alpha$, $\quad H = \frac{1}{\tan\alpha}\left(h\ \tan\alpha - \frac{2h}{3}\tan\alpha\right) = \frac{1}{\tan\alpha}\left(\frac{h\tan\alpha}{3}\right) = \frac{h}{3}$

Thus, the height of the cylinder is one-third the height of the cone when the volume of the cylinder is the greatest.

Now, the maximum volume of the cylinder can be obtained as:

$$V = \pi\left(\frac{2h}{3}\tan\alpha\right)^2\left(\frac{h}{3}\right) = \pi\left(\frac{4h^2}{3}tan^2\,\alpha\right)\left(\frac{h}{3}\right) = \frac{4}{27}\pi h^3 tan^2\alpha$$

Hence, the given result is proved.

Question 16:

A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic mere per hour. Then the depth of the wheat is increasing at the rate of

(A) 1 m/h (B) 0.1 m/h (C) 1.1 m/h (D) 0.5 m/h

Answer 16:

Let r be the radius of the cylinder.

Then, volume (V) of the cylinder is given by,

$V = \pi(radius)^2 \times height$

$\Rightarrow V = \pi(10)^2 h = 100\pi h$

Differentiating with respect to time t, we have:

$$\frac{dV}{dt} = 100\pi\ \frac{dh}{dt}$$

The tank is being filled with wheat at the rate of 314 cubic metre per hour.

$$\Rightarrow \frac{dV}{dt} = 314\ m^3/h$$

Thus we have, $314 = 100\pi\ \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{314}{100(3.14)} = \frac{314}{314} = 1$

Hence, the depth of wheat is increasing at the rate of 1 m/h.

The correct answer is (A).