

# Mathematics

## (Chapter - 5) (Continuity and Differentiability) (Exercise 5.1) (Class - XII)

### Question 1:

Prove that the function  $f(x) = 5x - 3$  is continuous at  $x = 0$ , at  $x = -3$  and at  $x = 5$ .

#### Answer 1:

Given function  $f(x) = 5x - 3$

At  $x = 0$ ,  $f(0) = 5(0) - 3 = -3$

LHL =  $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (5x - 3) = -3$

RHL =  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (5x - 3) = -3$

Here, at  $x = 0$ , LHL = RHL =  $f(0) = -3$

Hence, the function  $f$  is continuous at  $x = 0$ .

At  $x = -3$ ,  $f(-3) = 5(-3) - 3 = -18$

LHL =  $\lim_{x \rightarrow -3^-} f(x) = \lim_{x \rightarrow -3^-} (5x - 3) = -18$

RHL =  $\lim_{x \rightarrow -3^+} f(x) = \lim_{x \rightarrow -3^+} (5x - 3) = -18$

Here, at  $x = -3$ , LHL = RHL =  $f(-3) = -18$

Hence, the function  $f$  is continuous at  $x = -3$ .

At  $x = 5$ ,  $f(5) = 5(5) - 3 = 22$

LHL =  $\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^-} (5x - 3) = 22$

RHL =  $\lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5^+} (5x - 3) = 22$

Here, at  $x = 5$ , LHL = RHL =  $f(5) = 22$

Hence, the function  $f$  is continuous at  $x = 5$ .

### Question 2:

Examine the continuity of the function  $f(x) = 2x^2 - 1$  at  $x = 3$ .

#### Answer 2:

Given function  $f(x) = 2x^2 - 1$ . At  $x = 3$ ,  $f(3) = 2(3)^2 - 1 = 17$

LHL =  $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (2x^2 - 1) = 17$

RHL =  $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (2x^2 - 1) = 17$

Here, at  $x = 3$ , LHL = RHL =  $f(3) = 17$

Hence, the function  $f$  is continuous at  $x = 3$ .

### Question 3:

Examine the following functions for continuity:

(a)  $f(x) = x - 5$

(b)  $f(x) = \frac{1}{x-5}, x \neq 5$

(c)  $f(x) = \frac{x^2-25}{x+5}, x \neq -5$

(d)  $f(x) = |x - 5|$

#### Answer 3:

(a) Given function  $f(x) = x - 5$

Let,  $k$  be any real number. At  $x = k$ ,  $f(k) = k - 5$

LHL =  $\lim_{x \rightarrow k^-} f(x) = \lim_{x \rightarrow k^-} (x - 5) = k - 5$

RHL =  $\lim_{x \rightarrow k^+} f(x) = \lim_{x \rightarrow k^+} (x - 5) = k - 5$

At,  $x = k$ , LHL = RHL =  $f(k) = k - 5$

Hence, the function  $f$  is continuous for all real numbers.

**(b)** Given function  $f(x) = \frac{1}{x-5}, x \neq 5$

Let,  $k$  ( $k \neq 5$ ) be any real number. At  $x = k, f(k) = \frac{1}{k-5}$

$$\text{LHL} = \lim_{x \rightarrow k^-} f(x) = \lim_{x \rightarrow k^-} \left( \frac{1}{x-5} \right) = \frac{1}{k-5}$$

$$\text{RHL} = \lim_{x \rightarrow k^+} f(x) = \lim_{x \rightarrow k^+} \left( \frac{1}{x-5} \right) = \frac{1}{k-5}$$

$$\text{At, } x = k, \text{LHL} = \text{RHL} = f(k) = \frac{1}{k-5}$$

Hence, the function  $f$  is continuous for all real numbers (except 5).

**(c)** Given function  $f(x) = \frac{x^2-25}{x+5}, x \neq -5$

Let,  $k$  ( $k \neq -5$ ) be any real number.

$$\text{At } x = k, f(k) = \frac{k^2-25}{k+5} = \frac{(k+5)(k-5)}{(k+5)} = (k-5)$$

$$\text{LHL} = \lim_{x \rightarrow k^-} f(x) = \lim_{x \rightarrow k^-} \left( \frac{x^2-25}{x+5} \right) = \lim_{x \rightarrow k^-} \left( \frac{(k+5)(k-5)}{(k+5)} \right) = k-5$$

$$\text{RHL} = \lim_{x \rightarrow k^+} f(x) = \lim_{x \rightarrow k^+} \left( \frac{x^2-25}{x+5} \right) = \lim_{x \rightarrow k^+} \left( \frac{(k+5)(k-5)}{(k+5)} \right) = k-5$$

$$\text{At, } x = k, \text{LHL} = \text{RHL} = f(k) = k-5$$

Hence, the function  $f$  is continuous for all real numbers (except - 5).

**(d)** Given function  $f(x) = |x-5| = \begin{cases} 5-x, & x < 5 \\ x-5, & x \geq 5 \end{cases}$

Let,  $k$  be any real number. According to question,  $k < 5$  or  $k = 5$  or  $k > 5$ .

**First case:** If,  $k < 5$ ,

$$f(k) = 5 - k \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (5 - x) = 5 - k, \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers less than 5.

**Second case:** If,  $k = 5$ ,

$$f(k) = k - 5 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (x - 5) = k - 5, \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous at  $x = 5$ .

**Third case:** If,  $k > 5$ ,

$$f(k) = k - 5 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (x - 5) = k - 5, \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers greater than 5.

Hence, the function  $f$  is continuous for all real numbers.

#### Question 4:

Prove that the function  $f(x) = x^n$ , is continuous at  $x = n$ , where  $n$  is a positive integer.

#### Answer 4:

Given function  $f(x) = x^n$ .

$$\text{At } x = n, f(n) = n^n$$

$$\lim_{x \rightarrow n} f(x) = \lim_{x \rightarrow n} (x^n) = n^n$$

$$\text{Here, at } x = n, \lim_{x \rightarrow n} f(x) = f(n) = n^n$$

Hence, the function  $f$  is continuous at  $x = n$ , where  $n$  is positive integer.

### Question 5:

Is the function  $f$  defined by  $f(x) = \begin{cases} x, & x \leq 1 \\ 5, & x > 1 \end{cases}$  continuous at  $x = 0$ ? At  $x = 1$ ? At  $x = 2$ ?

#### Answer 5:

Given function  $f(x) = \begin{cases} x, & x \leq 1 \\ 5, & x > 1 \end{cases}$

At  $x = 0$ ,  $f(0) = 0$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (x) = 0$$

Here,  $x = 0$ ,  $\lim_{x \rightarrow 0} f(x) = f(0) = 0$

Hence, the function  $f$  is discontinuous at  $x = 0$ .

At  $x = 1$ ,  $f(1) = 1$

$$\text{LHL} = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x) = 1$$

$$\text{RHL} = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (5) = 5$$

Here, at  $x = 1$ ,  $\text{LHL} \neq \text{RHL}$ . Hence, the function  $f$  is discontinuous at  $x = 1$ .

At  $x = 2$ ,  $f(2) = 5$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (5) = 5$$

Here, at  $x = 2$ ,  $\lim_{x \rightarrow 2} f(x) = f(2) = 5$

Hence, the function  $f$  is continuous at  $x = 2$ .

Find all points of discontinuity of  $f$ , where  $f$  is defined by

### Question 6:

$$f(x) = \begin{cases} 2x + 3, & \text{If } x \leq 2 \\ 2x - 3, & \text{If } x > 2 \end{cases}$$

#### Answer 6:

Let,  $k$  be any real number. According to question,  $k < 2$  or  $k = 2$  or  $k > 2$

First case: यदि,  $k < 2$ ,

$$f(k) = 2k + 3 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (2x + 3) = 2k + 3, \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers smaller than 2.

Second case: If,  $k = 2$ ,  $f(2) = 2k + 3$

$$\text{LHL} = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2x + 3) = 7$$

$$\text{RHL} = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (2x - 3) = 1$$

Here, at  $x = 2$ ,  $\text{LHL} \neq \text{RHL}$ . Hence, the function  $f$  is discontinuous at  $x = 2$ .

Third case: If,  $k > 2$ ,

$$f(k) = 2k - 3 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (2x - 3) = 2k - 3, \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Therefore, the function  $f$  is continuous for all real numbers greater than 2.

Hence, the function  $f$  is discontinuous only at  $x = 2$ .

### Question 7:

$$f(x) = \begin{cases} |x| + 3, & \text{If } x \leq -3 \\ -2x, & \text{If } -3 < x < 3 \\ 6x + 2, & \text{If } x \geq 3 \end{cases}$$

#### Answer 7:

Let,  $k$  be any real number. According to question,

$k < -3$  or  $k = -3$  or  $-3 < k < 3$  or  $k = 3$  or  $k > 3$

First case: If,  $k < -3$ ,

$$f(k) = -k + 3 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (-x + 3) = -k + 3. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers less than  $-3$ .

Second case: If,  $k = -3$ ,  $f(-3) = -(-3) + 3 = 6$

$$\text{LHL} = \lim_{x \rightarrow -3^-} f(x) = \lim_{x \rightarrow -3^-} (-x + 3) = -(-3) + 3 = 6$$

$$\text{RHL} = \lim_{x \rightarrow -3^+} f(x) = \lim_{x \rightarrow -3^+} (-2x) = -2(-3) = 6. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous at  $x = -3$ .

Third case: If,  $-3 < k < 3$ ,

$$f(k) = -2k \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (-2x) = -2k. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous at  $-3 < x < 3$ .

Fourth case: If  $k = 3$ ,

$$\text{LHL} = \lim_{x \rightarrow k^-} f(x) = \lim_{x \rightarrow k^-} (-2x) = -2k$$

$$\text{RHL} = \lim_{x \rightarrow k^+} f(x) = \lim_{x \rightarrow k^+} (6x + 2) = 6k + 2,$$

Here, at  $x = 3$ ,  $\text{LHL} \neq \text{RHL}$ . Hence, the function  $f$  is discontinuous at  $x = 3$ .

Fifth case: If,  $k > 3$ ,

$$f(k) = 6k + 2 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (6x + 2) = 6k + 2. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all numbers greater than 3.

Hence, the function  $f$  is discontinuous only at  $x = 3$ .

### Question 8:

$$f(x) = \begin{cases} \frac{|x|}{x}, & \text{If } x \neq 0 \\ 0, & \text{If } x = 0 \end{cases}$$

#### Answer 8:

After redefining the function  $f$ , we get

$$f(x) = \begin{cases} -\frac{x}{x} = -1, & \text{If } x < 0 \\ 0, & \text{If } x = 0 \\ \frac{x}{x} = 1, & \text{If } x > 0 \end{cases}$$

Let,  $k$  be any real number. According to question,  $k < 0$  or  $k = 0$  or  $k > 0$ .

First case: If,  $k < 0$ ,

$$f(k) = -\frac{k}{k} = -1 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} \left(-\frac{x}{x}\right) = -1. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers smaller than 0.

Second case: If,  $k = 0$ ,  $f(0) = 0$

$$\text{LHL} = \lim_{x \rightarrow k^-} f(x) = \lim_{x \rightarrow k^-} \left(-\frac{x}{x}\right) = -1 \quad \text{and} \quad \text{RHL} = \lim_{x \rightarrow k^+} f(x) = \lim_{x \rightarrow k^+} \left(\frac{x}{x}\right) = 1,$$

Here, at  $x = 0$ ,  $\text{LHL} \neq \text{RHL}$ . Hence, the function  $f$  is discontinuous at  $x = 0$ .

Third case: If,  $k > 0$ ,

$$f(k) = \frac{k}{k} = 1 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} \left(\frac{x}{x}\right) = 1. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers greater than 0.

Therefore, the function  $f$  is discontinuous only at  $x = 0$ .

### Question 9:

$$f(x) = \begin{cases} \frac{x}{|x|}, & \text{If } x < 0 \\ -1, & \text{If } x \geq 0 \end{cases}$$

#### Answer 9:

Redefining the function, we get

$$f(x) = \begin{cases} \frac{x}{|x|} = \frac{x}{-x} = -1, & \text{If } x < 0 \\ -1, & \text{If } x \geq 0 \end{cases}$$

Here,  $\lim_{x \rightarrow k} f(x) = f(k) = -1$ , where  $k$  is a real number.

Hence, the function  $f$  is continuous for all real numbers.

### Question 10:

$$f(x) = \begin{cases} x + 1, & \text{If } x \geq 1 \\ x^2 + 1, & \text{If } x < 1 \end{cases}$$

#### Answer 10:

Let,  $k$  be any real number. According to question,  $k < 1$  or  $k = 1$  or  $k > 1$

First case: If,  $k < 1$ ,

$f(k) = k^2 + 1$  and  $\lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (x^2 + 1) = k^2 + 1$ . Here,  $\lim_{x \rightarrow k} f(x) = f(k)$

Hence, the function  $f$  is continuous for all real numbers smaller than 1.

Second case: If,  $k = 1$ ,  $f(1) = 1 + 1 = 2$

LHL =  $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^2 + 1) = 1 + 1 = 2$

RHL =  $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x + 1) = 1 + 1 = 2$ ,

Here, at  $x = 1$ , LHL = RHL =  $f(1)$ . Hence, the function  $f$  is continuous at  $x = 1$ .

Third case: If,  $k > 1$ ,

$f(k) = k + 1$  and  $\lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (x + 1) = k + 1$ . Here,  $\lim_{x \rightarrow k} f(x) = f(k)$

Hence, the function  $f$  is continuous for all real numbers greater than 1.

Therefore, the function  $f$  is continuous for all real numbers.

### Question 11:

$$f(x) = \begin{cases} x^3 - 3, & \text{If } x \leq 2 \\ x^2 + 1, & \text{If } x > 2 \end{cases}$$

#### Answer 11:

Let,  $k$  be any real number. According to question,  $k < 2$  or  $k = 2$  or  $k > 2$

First case: If,  $k < 2$ ,

$f(k) = k^3 - 3$  and  $\lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (x^3 - 3) = k^3 - 3$ . Here,  $\lim_{x \rightarrow k} f(x) = f(k)$

Hence, the function  $f$  is continuous for all real numbers less than 2.

Second case: If,  $k = 2$ ,  $f(2) = 2^3 - 3 = 5$

LHL =  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^3 - 3) = 2^3 - 3 = 5$

RHL =  $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^2 + 1) = 2^2 + 1 = 5$ ,

Here, at  $x = 2$ , LHL = RHL =  $f(2)$

Hence, the function  $f$  is continuous at  $x = 2$ .

Third case: If,  $k > 2$ ,

$f(k) = k^2 + 1$  and  $\lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (x^2 + 1) = k^2 + 1$ . Here,  $\lim_{x \rightarrow k} f(x) = f(k)$

Hence, the function  $f$  is continuous for real numbers greater than 2.

Hence, the function  $f$  is continuous for all real numbers.

**Question 12:**

$$f(x) = \begin{cases} x^{10} - 1, & \text{If } x \leq 1 \\ x^2, & \text{If } x > 1 \end{cases}$$

**Answer 12:**

Let,  $k$  be any real number. According to question,  $k < 1$  or  $k = 1$  or  $k > 1$

First case: If,  $k < 1$ ,

$$f(k) = k^{10} - 1 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (x^{10} - 1) = k^{10} - 1. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers less than 1.

Second case: If,  $k = 1$ ,  $f(1) = 1^{10} - 1 = 0$

$$\text{LHL} = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^{10} - 1) = 0$$

$$\text{RHL} = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^2) = 1,$$

Here, at  $x = 1$ ,  $\text{LHL} \neq \text{RHL}$ . Hence, the function  $f$  is discontinuous at  $x = 1$ .

Third case: If,  $k > 1$ ,

$$f(k) = k^2 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (x^2) = k^2. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real values greater than 1.

Hence, the function  $f$  is discontinuous only at  $x = 1$ .

**Question 13:**

Is the function defined by  $f(x) = \begin{cases} x + 5, & \text{If } x \leq 1 \\ x - 5, & \text{If } x > 1 \end{cases}$  a continuous function?

**Answer 13:**

Let,  $k$  be any real number. According to question,  $k < 1$  or  $k = 1$  or  $k > 1$

First case: If,  $k < 1$ ,

$$f(k) = k + 5 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (x + 5) = k + 5. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers less than 1.

Second case: If,  $k = 1$ ,  $f(1) = 1 + 5 = 6$

$$\text{LHL} = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x + 5) = 6$$

$$\text{RHL} = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x - 5) = -4,$$

Here, at  $x = 1$ ,  $\text{LHL} \neq \text{RHL}$ . Hence, the function  $f$  is discontinuous at  $x = 1$ .

Third case: If,  $k > 1$ ,

$$f(k) = k - 5 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (x - 5) = k - 5.$$

$$\text{Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers greater than 1.

Hence, the function  $f$  is discontinuous only at  $x = 1$ .

Discuss the continuity of the function  $f$ , where  $f$  is defined by:

**Question 14:**

$$f(x) = \begin{cases} 3, & \text{If } 0 \leq x \leq 1 \\ 4, & \text{If } 1 < x < 3 \\ 5, & \text{If } 3 \leq x \leq 10 \end{cases}$$

**Answer 14:**

Let,  $k$  be any real number. According to question,

$0 \leq k \leq 1$  or  $k = 1$  or  $1 < k < 3$  or  $k = 3$  or  $3 \leq k \leq 10$

**First case:** If,  $0 \leq k \leq 1$ ,

$$f(k) = 3 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (3) = 3. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for  $0 \leq x \leq 1$ .

**Second case:** If,  $k = 1$ ,  $f(1) = 3$

$$\text{LHL} = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3) = 3$$

$$\text{RHL} = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (4) = 4,$$

Here, at  $x = 1$ ,  $\text{LHL} \neq \text{RHL}$ . Hence, the function  $f$  is discontinuous at  $x = 1$ .

**Third case:** If,  $1 < k < 3$ ,

$$f(k) = 4 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (4) = 4. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for  $1 < x < 3$ .

**Fourth case:** If  $k = 3$ ,

$$\text{LHL} = \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (4) = 4 \quad \text{and} \quad \text{RHL} = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (5) = 5,$$

Here, at  $x = 3$ ,  $\text{LHL} \neq \text{RHL}$ . Hence, the function  $f$  is discontinuous at  $x = 3$ .

**Fifth case:** If,  $3 \leq k \leq 10$ ,

$$f(k) = 5 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (5) = 5. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for  $3 \leq x \leq 10$ .

Hence, the function  $f$  is discontinuous only at  $x = 1$  and  $x = 3$ .

### Question 15:

$$f(x) = \begin{cases} 2x, & \text{If } x < 0 \\ 0, & \text{If } 0 \leq x \leq 1 \\ 4x, & \text{If } x > 1 \end{cases}$$

### Answer 15:

Let,  $k$  be any real number. According to question,

$k < 0$  or  $k = 0$  or  $0 \leq k \leq 1$  or  $k = 1$  or  $k > 1$

**First case:** If,  $k < 0$ ,

$$f(k) = 2k \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (2x) = 2k. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers less than 0.

**Second case:** If,  $k = 0$ ,  $f(0) = 0$

$$\text{LHL} = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (2x) = 0$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (0) = 0. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous at  $x = 0$ .

**Third case:** If,  $0 \leq k \leq 1$ ,

$$f(k) = 0 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (0) = 0. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous at  $0 \leq x \leq 1$ .

**Fourth case:** If  $k = 1$ ,

$$\text{LHL} = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (0) = 0$$

$$\text{RHL} = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (4x) = 4,$$

Here, at  $x = 1$ ,  $\text{LHL} \neq \text{RHL}$ .

Hence, the function  $f$  is discontinuous at  $x = 1$ .

**Fifth case:** If,  $k > 1$ ,

$$f(k) = 4k \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (4x) = 4k.$$

$$\text{Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers greater than 1.

Therefore, the function  $f$  is discontinuous only at  $x = 1$ .

### Question 16:

$$f(x) = \begin{cases} -2, & \text{If } x \leq -1 \\ 2x, & \text{If } -1 < x \leq 1 \\ 2, & \text{If } x > 1 \end{cases}$$

#### Answer 16:

Let,  $k$  be any real number.

According to question,  $k < -1$  or  $k = -1$  or  $-1 < x \leq 1$  or  $k = 1$  or  $k > 1$

**First case:** If,  $k < -1$ ,

$$f(k) = -2 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (-2) = -2. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers less than  $-1$ .

**Second case:** If,  $k = -1$ ,  $f(-1) = -2$

$$\text{LHL} = \lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (-2) = -2$$

$$\text{RHL} = \lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (2x) = -2. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous at  $x = -1$ .

**Third case:** If,  $-1 < x \leq 1$ ,

$$f(k) = 2k \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (2x) = 2k. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous at  $-1 < x \leq 1$ .

**Fourth case:** If,  $k = 1$ ,

$$\text{LHL} = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x) = 2$$

$$\text{RHL} = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2) = 2. \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous at  $x = 1$ .

**Fifth case:** If,  $k > 1$ ,

$$f(k) = 2 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (2) = 2.$$

$$\text{Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers greater than 1.

Therefore, the function  $f$  is continuous for all real numbers.

### Question 17:

Find the relationship between  $a$  and  $b$  so that the function  $f$  defined by

$$f(x) = \begin{cases} ax + 1, & \text{If } x \leq 3 \\ bx + 3, & \text{If } x > 3 \end{cases}$$

is continuous at  $x = 3$ .

#### Answer 17:

Given that the function is continuous at  $x = 3$ . Therefore,  $\text{LHL} = \text{RHL} = f(3)$

$$\Rightarrow \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = f(3)$$

$$\Rightarrow \lim_{x \rightarrow 3^-} ax + 1 = \lim_{x \rightarrow 3^+} bx + 3 = 3a + 1$$

$$\Rightarrow 3a + 1 = 3b + 3 = 3a + 1 \Rightarrow 3a = 3b + 2 \Rightarrow a = b + \frac{2}{3}$$

### Question 18:

For what value of  $\lambda$  is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

continuous at  $x = 0$ ? What about continuity at  $x = 1$ ?

#### Answer 18:

Given that the function is continuous at  $x = 0$ . Therefore, LHL = RHL =  $f(0)$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} \lambda(x^2 - 2x) = \lim_{x \rightarrow 0^+} 4x + 1 = \lambda[(0)^2 - 2(0)]$$

$$\Rightarrow \lambda[(0)^2 - 2(0)] = 4(0) + 1 = \lambda(0)$$

$$\Rightarrow 0 \cdot \lambda = 1 \quad \Rightarrow \lambda = \frac{1}{0}$$

Hence, there is no real value of  $\lambda$  for which the given function be continuous.

If,  $x = 1$ ,

$$f(1) = 4(1) + 1 = 5 \text{ and } \lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} 4(1) + 1 = 5, \text{ Here, } \lim_{x \rightarrow 1} f(x) = f(1)$$

Hence, the function  $f$  is continuous for all real values of  $\lambda$ .

### Question 19:

Show that the function defined by  $g(x) = x - [x]$  is discontinuous at all integral points. Here  $[x]$  denotes the greatest integer less than or equal to  $x$ .

#### Answer 19:

Let,  $k$  be any integer.

$$\text{LHL} = \lim_{x \rightarrow k^-} f(x) = \lim_{x \rightarrow k^-} x - [x] = k - (k - 1) = 1$$

$$\text{RHL} = \lim_{x \rightarrow k^+} f(x) = \lim_{x \rightarrow k^+} x - [x] = k - (k) = 0,$$

Here, at  $x = k$ , LHL  $\neq$  RHL. Hence, the function  $f$  is discontinuous for all integers.

### Question 20:

Is the function defined by  $f(x) = x^2 - \sin x + 5$  continuous at  $x = \pi$ .

#### Answer 20:

Given function:  $f(x) = x^2 - \sin x + 5$ ,

$$\text{At } x = \pi, f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$$

$$\lim_{x \rightarrow \pi} f(x) = \lim_{x \rightarrow \pi} x^2 - \sin x + 5 = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$$

$$\text{Here, at } x = \pi, \lim_{x \rightarrow \pi} f(x) = f(\pi) = \pi^2 + 5$$

Hence, the function  $f$  is continuous at  $x = \pi$ .

### Question 21:

Discuss the continuity of the following functions:

(a)  $f(x) = \sin x + \cos x$

(b)  $f(x) = \sin x - \cos x$

(c)  $f(x) = \sin x \cdot \cos x$

#### Answer 21:

Let,  $g(x) = \sin x$

Let,  $k$  be any real number. At  $x = k$ ,  $g(k) = \sin k$

$$\text{LHL} = \lim_{x \rightarrow k^-} g(x) = \lim_{x \rightarrow k^-} \sin x = \lim_{h \rightarrow 0} \sin(k - h) = \lim_{h \rightarrow 0} \sin k \cos h - \cos k \sin h = \sin k$$

$$\text{RHL} = \lim_{x \rightarrow k^+} g(x) = \lim_{x \rightarrow k^+} \sin x = \lim_{h \rightarrow 0} \sin(k + h) = \lim_{h \rightarrow 0} \sin k \cos h + \cos k \sin h = \sin k$$

Here, at  $x = k$ , LHL = RHL =  $g(k)$ .

Hence, the function  $g$  is continuous for all real numbers.

Let,  $h(x) = \cos x$

Let,  $k$  be any real number. At  $x = k$ ,  $h(k) = \cos k$

$$\text{LHL} = \lim_{x \rightarrow k^-} h(x) = \lim_{x \rightarrow k^-} \cos x = \lim_{h \rightarrow 0} \cos(k - h) = \lim_{h \rightarrow 0} \cos k \cos h + \sin k \sin h = \cos k$$

$$\text{RHL} = \lim_{x \rightarrow k^+} h(x) = \lim_{x \rightarrow k^+} \cos x = \lim_{h \rightarrow 0} \cos(k + h) = \lim_{h \rightarrow 0} \cos k \cos h - \sin k \sin h = \cos k$$

Here, at  $x = k$ ,  $\text{LHL} = \text{RHL} = h(k)$ .

Hence, the function  $h$  is continuous for all real numbers.

We know that if  $g$  and  $h$  are two continuous functions, then the functions  $g + h$ ,  $g - h$  and  $gh$  also be a continuous functions.

Hence, **(a)**  $f(x) = \sin x + \cos x$  **(b)**  $f(x) = \sin x - \cos x$  and **(c)**  $f(x) = \sin x \cdot \cos x$  are continuous functions.

### Question 22:

Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

#### Answer 22:

Let  $g(x) = \sin x$

Let,  $k$  be any real number. At  $x = k$ ,  $g(k) = \sin k$

$$\text{LHL} = \lim_{x \rightarrow k^-} g(x) = \lim_{x \rightarrow k^-} \sin x = \lim_{h \rightarrow 0} \sin(k - h) = \lim_{h \rightarrow 0} \sin k \cos h - \cos k \sin h = \sin k$$

$$\text{RHL} = \lim_{x \rightarrow k^+} g(x) = \lim_{x \rightarrow k^+} \sin x = \lim_{h \rightarrow 0} \sin(k + h) = \lim_{h \rightarrow 0} \sin k \cos h + \cos k \sin h = \sin k$$

Here, at  $x = k$ ,  $\text{LHL} = \text{RHL} = g(k)$ .

Hence, the function  $g$  is continuous for all real numbers.

Let  $h(x) = \cos x$

Let,  $k$  be any real number. At  $x = k$ ,  $h(k) = \cos k$

$$\text{LHL} = \lim_{x \rightarrow k^-} h(x) = \lim_{x \rightarrow k^-} \cos x = \lim_{h \rightarrow 0} \cos(k - h) = \lim_{h \rightarrow 0} \cos k \cos h + \sin k \sin h = \cos k$$

$$\text{RHL} = \lim_{x \rightarrow k^+} h(x) = \lim_{x \rightarrow k^+} \cos x = \lim_{h \rightarrow 0} \cos(k + h) = \lim_{h \rightarrow 0} \cos k \cos h - \sin k \sin h = \cos k$$

Here, at  $x = k$ ,  $\text{LHL} = \text{RHL} = h(k)$ .

Hence, the function  $h$  is continuous for all real numbers.

We know that if  $g$  and  $h$  are two continuous functions, then the functions  $\frac{g}{h}$ ,  $h \neq 0$ ,  $\frac{1}{h}$ ,  $h \neq 0$  and  $\frac{1}{g}$ ,  $g \neq 0$  be continuous functions.

Therefore,  $\text{cosec } x = \frac{1}{\sin x}$ ,  $\sin x \neq 0$  is continuous  $\Rightarrow x \neq n\pi$  ( $n \in Z$ ) is continuous.

Hence,  $\text{cosec } x$  is continuous except  $x = n\pi$  ( $n \in Z$ ).

$\text{sec } x = \frac{1}{\cos x}$ ,  $\cos x \neq 0$  is continuous.  $\Rightarrow x \neq \frac{(2n+1)\pi}{2}$  ( $n \in Z$ ) is continuous.

Hence,  $\text{sec } x$  is continuous except  $x = \frac{(2n+1)\pi}{2}$  ( $n \in Z$ ).

$\text{cot } x = \frac{\cos x}{\sin x}$ ,  $\sin x \neq 0$  is continuous.  $\Rightarrow x \neq n\pi$  ( $n \in Z$ ) is continuous.

Hence,  $\text{cot } x$  is continuous except  $x = n\pi$  ( $n \in Z$ ).

### Question 23:

Find all points of discontinuity of  $f$ , where

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{If } x < 0 \\ x + 1, & \text{If } x \geq 0 \end{cases}$$

#### Answer 23:

Let,  $k$  be any real number. According to question,  $k < 0$  or  $k = 0$  or  $k > 0$

First case: If,  $k < 0$ ,

$$f(k) = \frac{\sin k}{k} \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} \left( \frac{\sin x}{x} \right) = \frac{\sin k}{k}, \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers less than 0.

**Second case:** If,  $k = 0$ ,  $f(0) = 0 + 1 = 1$

$$\text{LHL} = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x + 1) = 0 + 1 = 1$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x + 1) = 0 + 1 = 1,$$

Here, at  $x = 0$ ,  $\text{LHL} = \text{RHL} = f(0)$ .

Hence, the function  $f$  is continuous at  $x = 0$ .

**Third case:** If,  $k > 0$ ,

$$f(k) = k + 1 \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (x + 1) = k + 1, \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for all real numbers greater than 0.

Therefore, the function  $f$  is continuous for all real numbers.

### Question 24:

Determine if  $f$  defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{If } x \neq 0 \\ 0, & \text{If } x = 0 \end{cases}$$

is a continuous function?

#### Answer 24:

Let,  $k$  be any real number. According to question,  $k \neq 0$  or  $k = 0$

**First case:** If,  $k \neq 0$ ,

$$f(k) = k^2 \sin \frac{1}{k} \text{ and } \lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} \left( x^2 \sin \frac{1}{x} \right) = k^2 \sin \frac{1}{k}, \text{ Here, } \lim_{x \rightarrow k} f(x) = f(k)$$

Hence, the function  $f$  is continuous for  $k \neq 0$ .

**Second case:** If,  $k = 0$ ,  $f(0) = 0$

$$\text{LHL} = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left( x^2 \sin \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left( x^2 \sin \frac{1}{x} \right)$$

$$\text{We know that, } -1 \leq \sin \frac{1}{x} \leq 1, \quad x \neq 0 \quad \Rightarrow \quad -x^2 \leq \sin \frac{1}{x} \leq x^2$$

$$\Rightarrow \lim_{x \rightarrow 0} (-x^2) \leq \lim_{x \rightarrow 0} \sin \frac{1}{x} \leq \lim_{x \rightarrow 0} x^2$$

$$\Rightarrow 0 \leq \lim_{x \rightarrow 0} \sin \frac{1}{x} \leq 0 \quad \Rightarrow \quad \lim_{x \rightarrow 0} \sin \frac{1}{x} = 0 \quad \Rightarrow \quad \lim_{x \rightarrow 0^-} x^2 \sin \frac{1}{x} = 0 \quad \Rightarrow \quad \lim_{x \rightarrow 0^-} f(x) = 0$$

$$\text{Similarly, } \text{RHL} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left( x^2 \sin \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left( x^2 \sin \frac{1}{x} \right) = 0,$$

Here, at  $x = 0$ ,  $\text{LHL} = \text{RHL} = f(0)$

Hence, at  $x = 0$ ,  $f$  is continuous.

Hence, the function  $f$  is continuous for all real numbers.

### Question 25:

Examine the continuity of  $f$ , where  $f$  is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{If } x \neq 0 \\ -1, & \text{If } x = 0 \end{cases}$$

#### Answer 25:

Let,  $k$  be any real number. According to question,  $k \neq 0$  or  $k = 0$

**First case:** If,  $k \neq 0$ ,  $f(0) = 0 - 1 = -1$

$$\text{LHL} = \lim_{k \rightarrow 0^-} f(x) = \lim_{k \rightarrow 0^-} (\sin x - \cos x) = 0 - 1 = -1$$

$$\text{RHL} = \lim_{k \rightarrow 0^+} f(x) = \lim_{k \rightarrow 0^+} (\sin x - \cos x) = 0 - 1 = -1,$$

Hence, at  $x \neq 0$ ,  $\text{LHL} = \text{RHL} = f(x)$

Hence, the function  $f$  is continuous at  $x \neq 0$ .

Second case: If,  $k = 0$ ,  $f(k) = -1$

and  $\lim_{x \rightarrow k} f(x) = \lim_{x \rightarrow k} (-1) = -1$ , Here,  $\lim_{x \rightarrow k} f(x) = f(k)$

Hence, the function  $f$  is continuous at  $x = 0$ .

Therefore, the function  $f$  is continuous for all real numbers.

Find the values of  $k$  so that the function  $f$  is continuous at the indicated point in exercises 26 to 29.

### Question 26:

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{If } x \neq \frac{\pi}{2} \\ 3, & \text{If } x = \frac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

### Answer 26:

Given that the function is continuous at  $x = \frac{\pi}{2}$ . Therefore, LHL = RHL =  $f\left(\frac{\pi}{2}\right)$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = \lim_{x \rightarrow \frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{k \cos x}{\pi - 2x} = \lim_{x \rightarrow \frac{\pi}{2}^+} \frac{k \cos x}{\pi - 2x} = 3 \Rightarrow \lim_{h \rightarrow 0} \frac{k \cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h \rightarrow 0} \frac{k \cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = 3$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{k \sin h}{2h} = \lim_{h \rightarrow 0} \frac{-k \sin h}{-2h} = 3$$

$$\Rightarrow \frac{k}{2} = \frac{k}{2} = 3 \quad \left[ \because \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right]$$

$$\Rightarrow k = 6$$

### Question 27:

$$f(x) = \begin{cases} kx^2, & \text{If } x \leq 2 \\ 3, & \text{If } x > 2 \end{cases} \quad \text{at } x = 2$$

### Answer 27:

Given that the function is continuous at  $x = 2$ .

Therefore, LHL = RHL =  $f(2)$

$$\Rightarrow \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x \rightarrow 2^-} kx^2 = \lim_{x \rightarrow 2^+} 3 = k(2)^2$$

$$\Rightarrow 4k = 3 = 4k \Rightarrow k = \frac{3}{4}$$

### Question 28:

$$f(x) = \begin{cases} kx + 1, & \text{If } x \leq \pi \\ \cos x, & \text{If } x > \pi \end{cases} \quad \text{at } x = \pi$$

### Answer 28:

Given that the function is continuous at  $x = \pi$ .

Therefore, LHL = RHL =  $f(\pi)$

$$\Rightarrow \lim_{x \rightarrow \pi^-} f(x) = \lim_{x \rightarrow \pi^+} f(x) = f(\pi)$$

$$\Rightarrow \lim_{x \rightarrow \pi^-} kx + 1 = \lim_{x \rightarrow \pi^+} \cos x = k(\pi) + 1$$

$$\Rightarrow k(\pi) + 1 = \cos \pi = k\pi + 1 \Rightarrow k\pi + 1 = -1 = k\pi + 1$$

$$\Rightarrow \pi k = -2 \Rightarrow k = -\frac{2}{\pi}$$

**Question 29:**

$$f(x) = \begin{cases} kx + 1, & \text{If } x \leq 5 \\ 3x - 5, & \text{If } x > 5 \end{cases} \text{ at } x = 5$$

**Answer 29:**

Given that the function is continuous at  $x = 5$ .

Therefore, LHL = RHL =  $f(5)$

$$\Rightarrow \lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^+} f(x) = f(5)$$

$$\Rightarrow \lim_{x \rightarrow 5^-} kx + 1 = \lim_{x \rightarrow 5^+} 3x - 5 = 5k + 1$$

$$\Rightarrow 5k + 1 = 15 - 5 = 5k + 1$$

$$\Rightarrow 5k = 9 \quad \Rightarrow k = \frac{9}{5}$$

**Question 30:**

Find the values of  $a$  and  $b$  such that the function defined by

$$f(x) = \begin{cases} 5, & \text{If } x \leq 2 \\ ax + b, & \text{If } 2 < x < 10 \\ 21, & \text{If } x \geq 10 \end{cases}$$

is a continuous function.

**Answer 30:**

Given that the function is continuous at  $x = 2$ . Therefore, LHL = RHL =  $f(2)$

$$\Rightarrow \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x \rightarrow 2^-} 5 = \lim_{x \rightarrow 2^+} ax + b = 5$$

$$\Rightarrow 2a + b = 5 \quad \dots (1)$$

Given that the function is continuous at  $x = 10$ . Therefore, LHL = RHL =  $f(10)$

$$\Rightarrow \lim_{x \rightarrow 10^-} f(x) = \lim_{x \rightarrow 10^+} f(x) = f(10)$$

$$\Rightarrow \lim_{x \rightarrow 10^-} ax + b = \lim_{x \rightarrow 10^+} 21 = 21$$

$$\Rightarrow 10a + b = 21 \quad \dots (2)$$

Solving the equation (1) and (2), we get

$$a = 2 \quad b = 1$$

**Question 31:**

Show that the function defined by  $f(x) = \cos(x^2)$  is a continuous function.

**Answer 31:**

Assuming that the functions are well defined for all real numbers, we can write the given function  $f$  in the combination of  $g$  and  $h$  ( $f = goh$ ). Where,  $g(x) = \cos x$  and  $h(x) = x^2$ . If  $g$  and  $h$  both are continuous function then  $f$  also be continuous.

$$[\because goh(x) = g(h(x)) = g(x^2) = \cos(x^2)]$$

Function  $g(x) = \cos x$

Let,  $k$  be any real number. At  $x = k$ ,  $g(k) = \cos k$

$$\lim_{x \rightarrow k} g(x) = \lim_{x \rightarrow k} \cos x = \lim_{h \rightarrow 0} \cos(k + h) = \lim_{h \rightarrow 0} \cos k \cos h - \sin k \sin h = \cos k$$

Here,  $\lim_{x \rightarrow k} g(x) = g(k)$ , Hence, the function  $g$  is continuous for all real numbers.

Function  $h(x) = x^2$

Let,  $k$  be any real number. At  $x = k$ ,  $h(k) = k^2$

$$\lim_{x \rightarrow k} h(x) = \lim_{x \rightarrow k} x^2 = k^2$$

Here,  $\lim_{x \rightarrow k} h(x) = h(k)$ , Hence, the function  $h$  is continuous for all real numbers.

Therefore,  $g$  and  $h$  both are continuous function. Hence,  $f$  is continuous.

### Question 32:

Show that the function defined by  $f(x) = |\cos x|$  is a continuous function.

#### Answer 32:

Assuming that the functions are well defined for all real numbers, we can write the given function  $f$  in the combination of  $g$  and  $h$  ( $f = goh$ ). Where,  $g(x) = |x|$  and  $h(x) = \cos x$ . If  $g$  and  $h$  both are continuous function then  $f$  also be continuous.

$$[\because goh(x) = g(h(x)) = g(\cos x) = |\cos x|]$$

$$\text{Function } g(x) = |x|$$

Rearranging the function  $g$ , we get

$$g(x) = \begin{cases} -x, & \text{If } x < 0 \\ x, & \text{If } x \geq 0 \end{cases}$$

Let,  $k$  be any real number. According to question,  $k < 0$  or  $k = 0$  or  $k > 0$

**First case:** If,  $k < 0$ ,

$$g(k) = 0 \text{ and } \lim_{x \rightarrow k} g(x) = \lim_{x \rightarrow k} (-x) = 0, \text{ here, } \lim_{x \rightarrow k} g(x) = g(k)$$

Hence, the function  $g$  is continuous for all real numbers less than 0.

**Second case:** If,  $k = 0$ ,  $g(0) = 0 + 1 = 1$

$$\text{LHL} = \lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (x) = 0,$$

Here, at  $x = 0$ , LHL = RHL =  $g(0)$

Hence, the function  $g$  is continuous at  $x = 0$ .

**Third case:** If,  $k > 0$ ,

$$g(k) = 0 \text{ and } \lim_{x \rightarrow k} g(x) = \lim_{x \rightarrow k} (x) = 0, \text{ Here, } \lim_{x \rightarrow k} g(x) = g(k)$$

Hence, the function  $g$  is continuous for all real numbers greater than 0.

Hence, the function  $g$  is continuous for all real numbers.

Function  $h(x) = \cos x$

Let,  $k$  be any real number. At  $x = k$ ,  $h(k) = \cos k$

$$\lim_{x \rightarrow k} h(x) = \lim_{x \rightarrow k} \cos x = \cos k$$

Here,  $\lim_{x \rightarrow k} h(x) = h(k)$ , Hence, the function  $h$  is continuous for all real numbers.

Therefore,  $g$  and  $h$  both are continuous function. Hence,  $f$  is continuous.

### Question 33:

Examine that  $\sin |x|$  is a continuous function.

#### Answer 33:

Assuming that the functions are well defined for all real numbers, we can write the given function  $f$  in the combination of  $g$  and  $h$  ( $f = hog$ ). Where,  $h(x) = \sin x$  and  $g(x) = |x|$ . If  $g$  and  $h$  both are continuous function then  $f$  also be continuous.

$$[\because hog(x) = h(g(x)) = h(|x|) = \sin|x|]$$

$$\text{Function } h(x) = \sin x$$

Let,  $k$  be any real number. At  $x = k$ ,  $h(k) = \sin k$

$$\lim_{x \rightarrow k} h(x) = \lim_{x \rightarrow k} \sin x = \sin k$$

Here,  $\lim_{x \rightarrow k} h(x) = h(k)$ , Hence, the function  $h$  is continuous for all real numbers.

$$\text{Function } g(x) = |x|$$

Redefining the function  $g$ , we get

$$g(x) = \begin{cases} -x, & \text{If } x < 0 \\ x, & \text{If } x \geq 0 \end{cases}$$

Let,  $k$  be any real number. According to question,  $k < 0$  or  $k = 0$  or  $k > 0$

**First case:** If,  $k < 0$ ,

$$g(k) = 0 \text{ and } \lim_{x \rightarrow k} g(x) = \lim_{x \rightarrow k} (-x) = 0, \text{ Here, } \lim_{x \rightarrow k} g(x) = g(k)$$

Hence, the function  $g$  is continuous for all real numbers less than 0.

**Second case:** If,  $k = 0$ ,  $g(0) = 0 + 1 = 1$

$$\text{LHL} = \lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (x) = 0,$$

Here, at  $x = 0$ ,  $\text{LHL} = \text{RHL} = g(0)$

Hence, at  $x = 0$ , the function  $g$  is continuous.

**Third case:** If,  $k > 0$ ,

$$g(k) = 0 \text{ and } \lim_{x \rightarrow k} g(x) = \lim_{x \rightarrow k} (x) = 0, \text{ Here, } \lim_{x \rightarrow k} g(x) = g(k)$$

Hence, the function  $g$  is continuous for all real numbers greater than 0.

Hence, the function  $g$  is continuous for all real numbers.

Therefore,  $g$  and  $h$  both are continuous function. Hence,  $f$  is continuous.

### Question 34:

Find all the points of discontinuity of  $f$  defined by  $f(x) = |x| - |x + 1|$ .

#### Answer 34:

Assuming that the functions are well defined for all real numbers, we can write the given function  $f$  in the combination of  $g$  and  $h$  ( $f = g - h$ ), where,  $g(x) = |x|$  and  $h(x) = |x + 1|$ . If  $g$  and  $h$  both are continuous function then  $f$  also be continuous.

Function  $g(x) = |x|$

Redefining the function  $g$ , we get,

$$g(x) = \begin{cases} -x, & \text{If } x < 0 \\ x, & \text{If } x \geq 0 \end{cases}$$

Let,  $k$  be any real number. According to question,  $k < 0$  or  $k = 0$  or  $k > 0$

**First case:** If,  $k < 0$ ,

$$g(k) = 0 \text{ and } \lim_{x \rightarrow k} g(x) = \lim_{x \rightarrow k} (-x) = 0, \text{ Here, } \lim_{x \rightarrow k} g(x) = g(k)$$

Hence, the function  $g$  is continuous for all real numbers less than 0.

**Second case:** If,  $k = 0$ ,  $g(0) = 0 + 1 = 1$

$$\text{LHL} = \lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} (-x) = 0 \quad \text{and} \quad \text{RHL} = \lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} (x) = 0,$$

Here, at  $x = 0$ ,  $\text{LHL} = \text{RHL} = g(0)$

Hence, the function  $g$  is continuous at  $x = 0$ .

**Third case:** If,  $k > 0$ ,

$$g(k) = 0 \text{ and } \lim_{x \rightarrow k} g(x) = \lim_{x \rightarrow k} (x) = 0, \text{ Here, } \lim_{x \rightarrow k} g(x) = g(k)$$

Hence, the function  $g$  is continuous for all real numbers more than 0.

Hence, the function  $g$  is continuous for all real numbers.

Function  $h(x) = |x + 1|$

Redefining the function  $h$ , we get

$$h(x) = \begin{cases} -(x + 1), & \text{If } x < -1 \\ x + 1, & \text{If } x \geq -1 \end{cases}$$

Let,  $k$  be any real number. According to question,  $k < -1$  or  $k = -1$  or  $k > -1$

**First case:** If,  $k < -1$ ,

$$h(k) = -(k + 1) \text{ and } \lim_{x \rightarrow k} h(x) = \lim_{x \rightarrow k} -(k + 1) = -(k + 1), \text{ Here, } \lim_{x \rightarrow k} h(x) = h(k)$$

Hence, the function  $g$  is continuous for all real numbers less than  $-1$ .

**Second case:** If,  $k = -1$ ,  $h(-1) = -1 + 1 = 0$

$$\text{LHL} = \lim_{x \rightarrow -1^-} h(x) = \lim_{x \rightarrow -1^-} -(-1 + 1) = 0$$

$$\text{RHL} = \lim_{x \rightarrow -1^+} h(x) = \lim_{x \rightarrow -1^+} (x + 1) = -1 + 1 = 0,$$

Here, at  $x = -1$ ,  $\text{LHL} = \text{RHL} = h(-1)$

Hence, the function  $h$  is continuous at  $x = -1$ .

**Third case:** If,  $k > -1$ ,

$$h(k) = k + 1 \text{ and } \lim_{x \rightarrow k} h(x) = \lim_{x \rightarrow k} (k + 1) = k + 1, \text{ Here, } \lim_{x \rightarrow k} h(x) = h(k)$$

Hence, the function  $g$  is continuous for all real numbers greater than  $-1$ .

Hence, the function  $h$  is continuous for all real numbers.

Therefore,  $g$  and  $h$  both are continuous function. Hence,  $f$  is continuous.



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# Mathematics

(Chapter - 5) (Continuity and Differentiability) (Exercise 5.2)  
(Class - XII)

Differentiate the functions with respect to  $x$  in Exercises 1 to 8.

## Question 1:

$$\sin(x^2 + 5)$$

### Answer 1:

$$\text{Let } y = \sin(x^2 + 5)$$

Therefore,

$$\frac{dy}{dx} = \cos(x^2 + 5) \cdot \frac{d}{dx}(x^2 + 5) = \cos(x^2 + 5) \cdot 2x$$

## Question 2:

$$\cos(\sin x)$$

### Answer 2:

$$\text{Let } y = \cos(\sin x)$$

$$\text{Therefore, } \frac{dy}{dx} = -\sin(\sin x) \cdot \frac{d}{dx}(\sin x) = -\sin(\sin x) \cdot \cos x$$

## Question 3:

$$\sin(ax + b)$$

### Answer 3:

$$\text{Let } y = \sin(ax + b)$$

Therefore,

$$\frac{dy}{dx} = \cos(ax + b) \cdot \frac{d}{dx}(ax + b) = \cos(ax + b) \cdot a$$

## Question 4:

$$\sec(\tan(\sqrt{x}))$$

### Answer 4:

$$\text{Let } y = \sec(\tan(\sqrt{x}))$$

$$\begin{aligned} \text{Therefore, } \frac{dy}{dx} &= \sec(\tan \sqrt{x}) \tan(\tan \sqrt{x}) \cdot \frac{d}{dx}(\tan \sqrt{x}) = \sec(\tan \sqrt{x}) \tan(\tan \sqrt{x}) \cdot \sec^2 \sqrt{x} \frac{d}{dx}(\sqrt{x}) \\ &= \sec(\tan \sqrt{x}) \tan(\tan \sqrt{x}) \cdot \sec^2 \sqrt{x} \left(\frac{1}{2\sqrt{x}}\right) \end{aligned}$$

## Question 5:

$$\frac{\sin(ax + b)}{\cos(cx + d)}$$

### Answer 5:

$$\text{Let, } y = \frac{\sin(ax + b)}{\cos(cx + d)}$$

$$\text{Therefore, } \frac{dy}{dx} = \frac{\cos(cx + d) \cdot \frac{d}{dx} \sin(ax + b) - \sin(ax + b) \cdot \frac{d}{dx} \cos(cx + d)}{[\cos(cx + d)]^2}$$

$$= \frac{\cos(cx + d) \cdot \sin(ax + b) \cdot \frac{d}{dx}(ax + b) - \sin(ax + b) \cdot [-\sin(cx + d) \cdot \frac{d}{dx}(cx + d)]}{\cos^2(cx + d)}$$

$$= \frac{\cos(cx + d) \cdot \sin(ax + b) \cdot a + \sin(ax + b) \cdot \sin(cx + d) \cdot c}{\cos^2(cx + d)}$$

**Question 6:**

$$\cos x^3 \cdot \sin^2(x^5)$$

**Answer 6:**

Let  $y = \cos x^3 \cdot \sin^2(x^5)$

Therefore,

$$\begin{aligned} \frac{dy}{dx} &= \cos x^3 \cdot \frac{d}{dx} \sin^2(x^5) + \sin^2(x^5) \cdot \frac{d}{dx} \cos x^3 \\ &= \cos x^3 \cdot 2 \sin x^5 \cos x^5 \cdot \frac{d}{dx} x^5 + \sin^2(x^5) [-\sin x^3] \cdot \frac{d}{dx} x^3 \\ &= \cos x^3 \cdot 2 \sin x^5 \cos x^5 \cdot 5x^4 - \sin^2(x^5) \sin x^3 \cdot 3x^2 \end{aligned}$$

**Question 7:**

$$2\sqrt{\cot(x^2)}$$

**Answer 7:**

Let  $y = 2\sqrt{\cot(x^2)}$

Therefore,

$$\begin{aligned} \frac{dy}{dx} &= 2 \cdot \frac{1}{2\sqrt{\cot(x^2)}} \cdot \frac{d}{dx} [\cot(x^2)] \\ &= \frac{1}{\sqrt{\cot(x^2)}} \cdot [-\operatorname{cosec} x^2] \cdot \frac{d}{dx} x^2 \\ &= \frac{1}{\sqrt{\cot(x^2)}} \cdot [-\operatorname{cosec} x^2] \cdot 2x \end{aligned}$$

**Question 8:**

$$\cos(\sqrt{x})$$

**Answer 8:**

Let  $y = \cos(\sqrt{x})$

Therefore,

$$\begin{aligned} \frac{dy}{dx} &= -\sin(\sqrt{x}) \cdot \frac{d}{dx} \sqrt{x} \\ &= -\sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \end{aligned}$$

**Question 9:**

Prove that the function  $f$  given by  $f(x) = |x - 1|, x \in \mathbf{R}$ , is not differentiable at  $x = 1$ .

**Answer 9:**

At  $x = 1$ ,

$$\begin{aligned} LHD &= \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \rightarrow 0} \frac{|1-h-1| - |1-1|}{-h} = \lim_{h \rightarrow 0} \frac{h}{-h} = -1 \\ RHD &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{|1+h-1| - |1-1|}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1 \end{aligned}$$

Here,  $LHD \neq RHD$ , therefore, the function  $f(x) = |x - 1|, x \in \mathbf{R}$ , is not differentiable at  $x = 1$ .

**Question 10:**

Prove that the greatest integer function defined by  $f(x) = [x]$ ,  $0 < x < 3$ , is not differentiable at  $x = 1$  and  $x = 2$ .

**Answer 10:**

At  $x = 1$ ,

$$LHD = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{[1-h] - [1]}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{0 - 1}{-h} = \infty$$

$$RHD = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[1+h] - [1]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1 - 1}{h} = 0$$

Here,  $LHD \neq RHD$ , therefore, the function  $f(x) = [x]$ ,  $0 < x < 3$ , is not differentiable at  $x = 1$ .

At  $x = 2$ ,

$$LHD = \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{[2-h] - [2]}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{1 - 2}{-h} = \infty$$

$$RHD = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[2+h] - [2]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2 - 2}{h} = 0$$

Here,  $LHD \neq RHD$ , therefore, the function  $f(x) = [x]$ ,  $0 < x < 3$ , is not differentiable at  $x = 2$ .



# Mathematics

## (Chapter - 5) (Continuity and Differentiability) (Exercise 5.3) (Class - XII)

Find  $\frac{dy}{dx}$  in the following:

### Question 1:

$$2x + 3y = \sin x$$

#### Answer 1:

$$2x + 3y = \sin x$$

Differentiating both sides with respect to  $x$ , we get

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx} \sin x$$

$$\Rightarrow 2 + 3 \frac{dy}{dx} = \cos x \quad \Rightarrow \frac{dy}{dx} = \frac{\cos x - 2}{3}$$

### Question 2:

$$2x + 3y = \sin y$$

#### Answer 2:

$$2x + 3y = \sin y$$

Differentiating both sides with respect to  $x$ , we get

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx} \sin y \quad \Rightarrow 2 + 3 \frac{dy}{dx} = \cos y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(\cos y - 3) = 2 \quad \Rightarrow \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

### Question 3:

$$ax + by^2 = \cos y$$

#### Answer 3:

$$ax + by^2 = \cos y$$

Differentiating both sides with respect to  $x$ , we get

$$\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx} \cos y$$

$$\Rightarrow a + 2by \frac{dy}{dx} = -\sin y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(2by + \sin y) = -a \quad \Rightarrow \frac{dy}{dx} = -\frac{a}{2by + \sin y}$$

### Question 4:

$$xy + y^2 = \tan x + y$$

#### Answer 4:

$$xy + y^2 = \tan x + y$$

Differentiating both sides with respect to  $x$ , we get

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx} \tan x + \frac{dy}{dx}$$

$$\Rightarrow x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(x + 2y - 1) = \sec^2 x - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1}$$

**Question 5:**

$$x^2 + xy + y^2 = 100$$

**Answer 5:**

$$x^2 + xy + y^2 = 100$$

Differentiating both sides with respect to  $x$ , we get

$$\frac{d}{dx}x^2 + \frac{d}{dx}(xy) + \frac{d}{dx}y^2 = \frac{d}{dx}(100)$$

$$\Rightarrow 2x + x \frac{dy}{dx} + y + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(x + 2y) = 2x + y \quad \Rightarrow \frac{dy}{dx} = \frac{2x + y}{x + 2y}$$

**Question 6:**

$$x^3 + x^2y + xy^2 + y^3 = 81$$

**Answer 6:**

$$x^3 + x^2y + xy^2 + y^3 = 81$$

Differentiating both sides with respect to  $x$ , we get

$$\frac{d}{dx}x^3 + \frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) + \frac{d}{dx}y^3 = \frac{d}{dx}81$$

$$\Rightarrow 3x^2 + x^2 \frac{dy}{dx} + y \cdot 2x + x \cdot 2y \frac{dy}{dx} + y^2 \cdot 1 + 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(x^2 + 2xy + 3y^2) = -(3x^2 + 2xy + y^2) \quad \Rightarrow \frac{dy}{dx} = -\frac{3x^2 + 2xy + y^2}{x^2 + 2xy + 3y^2}$$

**Question 7:**

$$\sin^2 y + \cos xy = k$$

**Answer 7:**

$$\sin^2 y + \cos xy = k$$

Differentiating both sides with respect to  $x$ , we get

$$\frac{d}{dx}\sin^2 y + \frac{d}{dx}\cos xy = \frac{d}{dx}k$$

$$\Rightarrow 2 \sin y \cos y \frac{dy}{dx} - \sin xy \left( x \frac{dy}{dx} + y \right) = 0 \Rightarrow \sin 2y \frac{dy}{dx} - x \sin xy \frac{dy}{dx} - y \sin xy = 0$$

$$\Rightarrow (\sin 2y - x \sin xy) \frac{dy}{dx} = y \sin xy \quad \Rightarrow \frac{dy}{dx} = \frac{y \sin xy}{\sin 2y - x \sin xy}$$

**Question 8:**

$$\sin^2 x + \cos^2 y = 1$$

**Answer 8:**

$$\sin^2 x + \cos^2 y = 1$$

Differentiating both sides with respect to  $x$ , we get

$$\frac{d}{dx}\sin^2 x + \frac{d}{dx}\cos^2 y = \frac{d}{dx}1$$

$$\Rightarrow 2 \sin x \cos x + 2 \cos y (-\sin y) \frac{dy}{dx} = 0$$

$$\Rightarrow \sin 2x - \sin 2y \frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

**Question 9:**

$$y = \sin^{-1} \left( \frac{2x}{1+x^2} \right)$$

**Answer 9:**

$$y = \sin^{-1} \left( \frac{2x}{1+x^2} \right)$$

Let,  $x = \tan \theta$

Therefore,  $y = \sin^{-1} \left( \frac{2 \tan \theta}{1 + \tan^2 \theta} \right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2 \tan^{-1} x \Rightarrow y = 2 \tan^{-1} x$

Differentiating both sides with respect to  $x$ , we get

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

**Question 10:**

$$y = \tan^{-1} \left( \frac{3x-x^3}{1-3x^2} \right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

**Answer 10:**

$$y = \tan^{-1} \left( \frac{3x-x^3}{1-3x^2} \right)$$

Let,  $x = \tan \theta$

Therefore,  $y = \tan^{-1} \left( \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right) = \tan^{-1}(\tan 3\theta) = 3\theta = 3 \tan^{-1} x \Rightarrow y = 3 \tan^{-1} x$

Differentiating both sides with respect to  $x$ , we get

$$\frac{dy}{dx} = \frac{3}{1+x^2}$$

**Question 11:**

$$y = \cos^{-1} \left( \frac{1-x^2}{1+x^2} \right), 0 < x < 1$$

**Answer 11:**

$$y = \cos^{-1} \left( \frac{1-x^2}{1+x^2} \right)$$

Let,  $x = \tan \theta$

Therefore,  $y = \cos^{-1} \left( \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) = \cos^{-1}(\cos 2\theta) = 2\theta = 2 \tan^{-1} x \Rightarrow y = 2 \tan^{-1} x$

Differentiating both sides with respect to  $x$ , we get

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

**Question 12:**

$$y = \sin^{-1} \left( \frac{1-x^2}{1+x^2} \right), 0 < x < 1$$

**Answer 12:**

$$y = \sin^{-1} \left( \frac{1-x^2}{1+x^2} \right)$$

Let,  $x = \tan \theta$ , Therefore,

$$y = \sin^{-1} \left( \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) = \sin^{-1}(\cos 2\theta) = \sin^{-1} \left\{ \sin \left( \frac{\pi}{2} - 2\theta \right) \right\} = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2 \tan^{-1} x$$

$$\Rightarrow y = \frac{\pi}{2} - 2 \tan^{-1} x$$

Differentiating both sides with respect to  $x$ , we get

$$\frac{dy}{dx} = 0 - \frac{2}{1+x^2} = -\frac{2}{1+x^2}$$

**Question 13:**

$$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$$

**Answer 13:**

$$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$$

Let,  $x = \tan \theta$

$$\text{Therefore, } y = \cos^{-1}\left(\frac{2 \tan \theta}{1+\tan^2 \theta}\right)$$

$$= \cos^{-1}(\sin 2\theta) = \cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 2\theta\right)\right\} = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2 \tan^{-1} x$$

$$\Rightarrow y = \frac{\pi}{2} - 2 \tan^{-1} x$$

Differentiating both sides with respect to  $x$ , we get

$$\frac{dy}{dx} = 0 - \frac{2}{1+x^2} = -\frac{2}{1+x^2}$$

**Question 14:**

$$y = \sin^{-1}(2x\sqrt{1-x^2}), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$$

**Answer 14:**

$$y = \sin^{-1}(2x\sqrt{1-x^2})$$

Let,  $x = \sin \theta$

$$\text{Therefore, } y = \sin^{-1}(2 \sin \theta \sqrt{1-\sin^2 \theta})$$

$$= \sin^{-1}(2 \sin \theta \cos \theta) = \sin^{-1}(\sin 2\theta) = 2\theta = 2 \sin^{-1} x$$

$$\Rightarrow y = 2 \sin^{-1} x$$

Differentiating both sides with respect to  $x$ , we get

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

**Question 15:**

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$$

**Answer 15:**

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$$

Let,  $x = \cos \theta$

$$\text{Therefore, } y = \sec^{-1}\left(\frac{1}{2\cos^2 \theta - 1}\right) = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right) = \sec^{-1}(\sec 2\theta) = 2\theta = 2 \cos^{-1} x$$

$$\Rightarrow y = 2 \cos^{-1} x$$

Differentiating both sides with respect to  $x$ , we get

$$\frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$$

# Mathematics

(Chapter - 5) (Continuity and Differentiability) (Exercise 5.4)  
(Class - XII)

Differentiate the following w.r.t.  $x$ :

## Question 1:

$$\frac{e^x}{\sin x}$$

### Answer 1:

Let  $y = \frac{e^x}{\sin x}$ , Therefore,

$$\frac{dy}{dx} = \frac{e^x \cdot \frac{d}{dx} \sin x - \sin x \cdot \frac{d}{dx} e^x}{\sin^2 x} = \frac{e^x \cdot \cos x - \sin x \cdot e^x}{\sin^2 x} = \frac{e^x(\cos x - \sin x)}{\sin^2 x}$$

## Question 2:

$$e^{\sin^{-1} x}$$

### Answer 2:

Let  $y = e^{\sin^{-1} x}$ , Therefore,

$$\frac{dy}{dx} = e^{\sin^{-1} x} \cdot \frac{d}{dx} \sin^{-1} x = e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} = \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$$

## Question 3:

$$e^{x^3}$$

### Answer 3:

Let  $y = e^{x^3}$ , Therefore,

$$\frac{dy}{dx} = e^{x^3} \cdot \frac{d}{dx} x^3 = e^{x^3} \cdot 3x^2 = 3x^2 e^{x^3}$$

## Question 4:

$$\sin(\tan^{-1} e^{-x})$$

### Answer 4:

Let  $y = \sin(\tan^{-1} e^{-x})$ , Therefore,

$$\begin{aligned} \frac{dy}{dx} &= \cos(\tan^{-1} e^{-x}) \cdot \frac{d}{dx} \tan^{-1} e^{-x} = \cos(\tan^{-1} e^{-x}) \cdot \frac{1}{1+(e^{-x})^2} \cdot \frac{d}{dx} e^{-x} \\ &= \cos(\tan^{-1} e^{-x}) \cdot \frac{1}{1+e^{-2x}} \cdot (-e^{-x}) = -\frac{e^{-x} \cos(\tan^{-1} e^{-x})}{1+e^{-2x}} \end{aligned}$$

## Question 5:

$$\log(\cos e^x)$$

### Answer 5:

Let  $y = \log(\cos e^x)$ , therefore,

$$\frac{dy}{dx} = \frac{1}{\cos e^x} \cdot \frac{d}{dx} \cos e^x = \frac{1}{\cos e^x} (-\sin e^x) \frac{d}{dx} e^x = -\tan e^x \cdot e^x$$

## Question 6:

$$e^x + e^{x^2} + \dots + e^{x^5}$$

### Answer 6:

Let  $y = e^x + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}$ , Therefore,

$$\begin{aligned} \frac{dy}{dx} &= e^x + e^{x^2} \frac{d}{dx} x^2 + e^{x^3} \frac{d}{dx} x^3 + e^{x^4} \frac{d}{dx} x^4 + e^{x^5} \frac{d}{dx} x^5 \\ &= e^x + e^{x^2} \cdot 2x + e^{x^3} \cdot 3x^2 + e^{x^4} \cdot 4x^3 + e^{x^5} \cdot 5x^4 \\ &= e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5} \end{aligned}$$

**Question 7:**

$$\sqrt{e^{\sqrt{x}}}, x > 0$$

**Answer 7:**

$$\text{Let } y = \sqrt{e^{\sqrt{x}}}$$

Therefore,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{e^{\sqrt{x}}}} \frac{d}{dx} e^{\sqrt{x}}$$

$$= \frac{1}{2\sqrt{e^{\sqrt{x}}}} \cdot e^{\sqrt{x}} \cdot \frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{e^{\sqrt{x}}}} \cdot e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{\sqrt{e^{\sqrt{x}}}}{4\sqrt{x}}$$

**Question 8:**

$$\log(\log x), x > 1$$

**Answer 8:**

$$\text{Let } y = \frac{e^x}{\sin x}$$

Therefore,

$$\frac{dy}{dx} = \frac{1}{\log x} \cdot \frac{d}{dx} \log x$$

$$= \frac{1}{\log x} \cdot \frac{1}{x} = \frac{1}{x \log x}$$

**Question 9:**

$$\frac{\cos x}{\log x}, x > 0$$

**Answer 9:**

$$\text{Let } y = \frac{\cos x}{\log x}$$

Therefore,

$$\frac{dy}{dx} = \frac{\log x \frac{d}{dx} \cos x - \cos x \frac{d}{dx} \log x}{(\log x)^2}$$

$$= \frac{\log x \cdot (-\sin x) - \cos x \cdot \frac{1}{x}}{(\log x)^2} = \frac{-(x \sin x \log x + \cos x)}{x(\log x)^2}$$

**Question 10:**

$$\cos(\log x + e^x), x > 0$$

**Answer 10:**

$$\text{Let } y = \cos(\log x + e^x)$$

Therefore,

$$\frac{dy}{dx} = -\sin(\log x + e^x) \cdot \frac{d}{dx} (\log x + e^x)$$

$$= -\sin(\log x + e^x) \cdot \left(\frac{1}{x} + e^x\right)$$

# Mathematics

## (Chapter - 5) (Continuity and Differentiability) (Exercise 5.5)

(Class - XII)

Differentiate the functions given in Exercises 1 to 11 w.r.t.  $x$ .

### Question 1:

$$\cos x \cdot \cos 2x \cdot \cos 3x$$

#### Answer 1:

Let  $y = \cos x \cdot \cos 2x \cdot \cos 3x$ , taking log on both the sides

$$\log y = \log \cos x + \log \cos 2x + \log \cos 3x$$

Therefore,

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{\cos x} \cdot \frac{d}{dx} \cos x + \frac{1}{\cos 2x} \cdot \frac{d}{dx} \cos 2x + \frac{1}{\cos 3x} \cdot \frac{d}{dx} \cos 3x$$

$$\Rightarrow \frac{dy}{dx} = y \left[ \frac{1}{\cos x} \cdot (-\sin x) + \frac{1}{\cos 2x} \cdot (-\sin 2x) \cdot 2 + \frac{1}{\cos 3x} \cdot (-\sin 3x) \cdot 3 \right]$$

$$\Rightarrow \frac{dy}{dx} = \cos x \cdot \cos 2x \cdot \cos 3x [-\tan x - 2 \tan 2x - 3 \tan 3x]$$

### Question 2:

$$\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

#### Answer 2:

Let  $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ , taking log on both the sides

$$\log y = \frac{1}{2} [\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)]$$

Therefore,

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left[ \frac{1}{(x-1)} + \frac{1}{(x-2)} - \frac{1}{(x-3)} - \frac{1}{(x-4)} - \frac{1}{(x-5)} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}} \left[ \frac{1}{(x-1)} + \frac{1}{(x-2)} - \frac{1}{(x-3)} - \frac{1}{(x-4)} - \frac{1}{(x-5)} \right]$$

### Question 3:

$$(\log x)^{\cos x}$$

#### Answer 3:

Let  $y = (\log x)^{\cos x}$ , taking log on both the sides

$$\log y = \log(\log x)^{\cos x} = \cos x \cdot \log \log x$$

Therefore,

$$\frac{1}{y} \frac{dy}{dx} = \cos x \cdot \frac{d}{dx} \log \log x + \log \log x \cdot \frac{d}{dx} \cos x$$

$$\Rightarrow \frac{dy}{dx} = y \left[ \cos x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log \log x \cdot (-\sin x) \right]$$

$$\Rightarrow \frac{dy}{dx} = (\log x)^{\cos x} \left[ \frac{\cos x - \sin x \log \log x}{x \log x} \right]$$

**Question 4:**

$$x^x - 2^{\sin x}$$

**Answer 4:**

Let  $u = x^x$  and  $v = 2^{\sin x}$  therefore,  $y = u - v$

Differentiating with respect to  $x$  on both sides

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \quad \dots (1)$$

Here,  $u = x^x$ , taking log on both the sides

$\log u = x \log x$ , therefore,

$$\frac{1}{u} \frac{du}{dx} = x \cdot \frac{d}{dx} \log x + \log x \cdot \frac{d}{dx} x = x \cdot \frac{1}{x} + \log x \cdot 1 = 1 + \log x$$

$$\frac{du}{dx} = u[1 + \log x] = x^x[1 + \log x] \quad \dots (2)$$

and,  $v = 2^{\sin x}$ , taking log on both the sides

$\log v = \sin x \log 2$ , therefore,

$$\frac{1}{v} \frac{dv}{dx} = \log 2 \cdot \frac{d}{dx} \sin x = \log 2 \cdot \cos x$$

$$\frac{dv}{dx} = v[\cos x \log 2] = 2^{\sin x}[\cos x \log 2] \quad \dots (3)$$

Putting the value of  $\frac{du}{dx}$  from (2) and  $\frac{dv}{dx}$  from (3) in equation (1), we get

$$\frac{dy}{dx} = x^x[1 + \log x] - 2^{\sin x}[\cos x \log 2]$$

**Question 5:**

$$(x + 3)^2 \cdot (x + 4)^3 \cdot (x + 5)^4$$

**Answer 5:**

Let  $y = (x + 3)^2 \cdot (x + 4)^3 \cdot (x + 5)^4$ , taking log on both the sides

$$\log y = 2 \log(x + 3) + 3 \log(x + 4) + 4 \log(x + 5)$$

Therefore,

$$\frac{1}{y} \frac{dy}{dx} = 2 \cdot \frac{1}{(x + 3)} + 3 \cdot \frac{1}{(x + 4)} + 4 \cdot \frac{1}{(x + 5)}$$

$$\Rightarrow \frac{dy}{dx} = y \left[ \frac{2(x + 4)(x + 5) + 3(x + 3)(x + 5) + 4(x + 3)(x + 4)}{(x + 3)(x + 4)(x + 5)} \right]$$

$$\Rightarrow \frac{dy}{dx} = y \left[ \frac{2(x^2 + 9x + 20) + 3(x^2 + 8x + 15) + 4(x^2 + 7x + 12)}{(x + 3)(x + 4)(x + 5)} \right]$$

$$\Rightarrow \frac{dy}{dx} = (x + 3)^2 \cdot (x + 4)^3 \cdot (x + 5)^4 \left[ \frac{9x^2 + 70x + 133}{(x + 3)(x + 4)(x + 5)} \right]$$

$$\Rightarrow \frac{dy}{dx} = (x + 3) \cdot (x + 4)^2 \cdot (x + 5)^3 (9x^2 + 70x + 133)$$

**Question 6:**

$$\left(x + \frac{1}{x}\right)^x + x^{\left(1 + \frac{1}{x}\right)}$$

**Answer 6:**

Let  $u = \left(x + \frac{1}{x}\right)^x$  and  $v = x^{\left(1 + \frac{1}{x}\right)}$ , therefore,  $y = u + v$

Differentiating with respect to  $x$ , we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots (1)$$

Here,  $u = \left(x + \frac{1}{x}\right)^x$ , taking log on both the sides

$\log u = x \log \left(x + \frac{1}{x}\right)$ , therefore,

$$\frac{1}{u} \frac{du}{dx} = x \cdot \frac{d}{dx} \log \left(x + \frac{1}{x}\right) + \log \left(x + \frac{1}{x}\right) \cdot \frac{d}{dx} x$$

$$= x \cdot \frac{1}{\left(x + \frac{1}{x}\right)} \cdot \left(1 - \frac{1}{x^2}\right) + \log \left(x + \frac{1}{x}\right) \cdot 1 = \frac{x^2}{x^2 + 1} \cdot \frac{x^2 - 1}{x^2} + \log \left(x + \frac{1}{x}\right)$$

$$\frac{du}{dx} = \left(x + \frac{1}{x}\right)^x \left[ \frac{x^2 - 1}{x^2 + 1} + \log \left(x + \frac{1}{x}\right) \right] \quad \dots (2)$$

and,  $v = x^{\left(1 + \frac{1}{x}\right)}$ , taking log on both the sides

$\log v = \left(1 + \frac{1}{x}\right) \log x$ , therefore,

$$\frac{1}{v} \frac{dv}{dx} = \left(1 + \frac{1}{x}\right) \cdot \frac{d}{dx} \log x + \log x \cdot \frac{d}{dx} \left(1 + \frac{1}{x}\right) = \left(1 + \frac{1}{x}\right) \cdot \frac{1}{x} + \log x \cdot \left(-\frac{1}{x^2}\right)$$

$$\frac{dv}{dx} = v \left[ \left(\frac{x^2 + 1}{x}\right) \cdot \frac{1}{x} - \frac{\log x}{x^2} \right] = x^{\left(1 + \frac{1}{x}\right)} \left[ \frac{x^2 + 1 - \log x}{x^2} \right] \quad \dots (3)$$

Putting the value of  $\frac{du}{dx}$  from (2) and  $\frac{dv}{dx}$  from (3) in equation (1), we get

$$\frac{dy}{dx} = \left(x + \frac{1}{x}\right)^x \left[ \frac{x^2 - 1}{x^2 + 1} + \log \left(x + \frac{1}{x}\right) \right] + x^{\left(1 + \frac{1}{x}\right)} \left[ \frac{x^2 + 1 - \log x}{x^2} \right]$$

**Question 7:**

$$(\log x)^x + x^{\log x}$$

**Answer 7:**

Let  $u = (\log x)^x$  and  $v = x^{\log x}$ , therefore,  $y = u + v$

Differentiating with respect to  $x$ , we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots (1)$$

Here,  $u = (\log x)^x$ , taking log on both the sides

$\log u = x \log \log x$ , therefore,

$$\frac{1}{u} \frac{du}{dx} = x \cdot \frac{d}{dx} \log \log x + \log \log x \cdot \frac{d}{dx} x$$

$$= x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log \log x \cdot 1 = \frac{1}{\log x} + \log \log x$$

$$\frac{du}{dx} = (\log x)^x \left[ \frac{1 + \log x \cdot \log \log x}{\log x} \right]$$

$$= (\log x)^{x-1} (1 + \log x \cdot \log \log x) \quad \dots (2)$$

and,  $v = x^{\log x}$ , taking log on both the sides

$\log v = \log x \log x$ , therefore,

$$\frac{1}{v} \frac{dv}{dx} = \log x \cdot \frac{d}{dx} \log x + \log x \cdot \frac{d}{dx} \log x$$

$$= \log x \cdot \frac{1}{x} + \log x \cdot \frac{1}{x}$$

$$\frac{dv}{dx} = v \left[ \frac{2 \log x}{x} \right] = x^{\log x} \left[ \frac{2 \log x}{x} \right] = x^{\log x - 1} (2 \log x) \quad \dots (3)$$

Putting the value of  $\frac{du}{dx}$  from (2) and  $\frac{dv}{dx}$  from (3) in equation (1), we get

$$\frac{dy}{dx} = (\log x)^{x-1} (1 + \log x \cdot \log \log x) + x^{\log x - 1} (2 \log x)$$

### Question 8:

$$(\sin x)^x + \sin^{-1} \sqrt{x}$$

### Answer 8:

Let  $u = (\sin x)^x$  and  $v = \sin^{-1} \sqrt{x}$ , therefore,  $y = u + v$

Differentiating with respect to  $x$ , we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots (1)$$

Here,  $u = (\sin x)^x$ , taking log on both the sides

$\log u = x \log \sin x$ , therefore,

$$\frac{1}{u} \frac{du}{dx} = x \cdot \frac{d}{dx} \log \sin x + \log \sin x \cdot \frac{d}{dx} x$$

$$= x \cdot \frac{1}{\sin x} \cdot \cos x + \log \sin x \cdot 1 = x \cot x + \log \sin x$$

$$\frac{du}{dx} = (\sin x)^x (x \cot x + \log \sin x) \quad \dots (2)$$

and,  $v = \sin^{-1} \sqrt{x}$ , therefore,

$$\frac{1}{v} \frac{dv}{dx} = \log x \cdot \frac{d}{dx} \log x + \log x \cdot \frac{d}{dx} \log x = \log x \cdot \frac{1}{x} + \log x \cdot \frac{1}{x}$$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{d}{dx} \sqrt{x} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x-x^2}} \quad \dots (3)$$

Putting the value of  $\frac{du}{dx}$  from (2) and  $\frac{dv}{dx}$  from (3) in equation (1), we get

$$\frac{dy}{dx} = (\sin x)^x (x \cot x + \log \sin x) + \frac{1}{2\sqrt{x-x^2}}$$

**Question 9:**

$$x^{\sin x} + (\sin x)^{\cos x}$$

**Answer 9:**

Let  $u = x^{\sin x}$  and  $v = (\sin x)^{\cos x}$  therefore,  $y = u + v$

Differentiating with respect to  $x$ , we get,  $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$  ... (1)

Here,  $u = x^{\sin x}$ , taking log on both the sides

$\log u = \sin x \log x$ , therefore,

$$\frac{1}{u} \frac{du}{dx} = \sin x \cdot \frac{d}{dx} \log x + \log x \cdot \frac{d}{dx} \sin x = \sin x \cdot \frac{1}{x} + \log x \cdot \cos x = \frac{\sin x}{x} + \log x \cos x$$

$$\frac{du}{dx} = x^{\sin x} \left[ \frac{\sin x}{x} + \log x \cos x \right] = x^{\sin x - 1} (\sin x + x \log x \cos x) \quad \dots (2)$$

and,  $v = (\sin x)^{\cos x}$ , taking log on both the sides, we get  $\log v = \cos x \log \sin x$ , therefore,

$$\frac{1}{v} \frac{dv}{dx} = \cos x \cdot \frac{d}{dx} \log \sin x + \log \sin x \cdot \frac{d}{dx} \cos x = \cos x \cdot \frac{1}{\sin x} \cos x + \log \sin x (-\sin x)$$

$$\frac{dv}{dx} = v [\cos x \cot x - \sin x \log \sin x] = (\sin x)^{\cos x} (\cos x \cot x - \sin x \log \sin x) \quad \dots (3)$$

Putting the value of  $\frac{du}{dx}$  from (2) and  $\frac{dv}{dx}$  from (3) in equation (1), we get

$$\frac{dy}{dx} = x^{\sin x - 1} (\sin x + x \log x \cos x) + (\sin x)^{\cos x} (\cos x \cot x - \sin x \log \sin x)$$

**Question 10:**

$$x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$$

**Answer 10:**

Let  $u = x^{x \cos x}$  and  $v = \frac{x^2 + 1}{x^2 - 1}$  therefore,  $y = u + v$

Differentiating with respect to  $x$ , we get  $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$  ... (1)

Here,  $u = x^{x \cos x}$ , taking log on both the sides, we get  $\log u = x \log x$ , therefore,

$$\frac{1}{u} \frac{du}{dx} = x \cos x \cdot \frac{d}{dx} \log x + \log x \cdot \frac{d}{dx} x \cos x = x \cos x \cdot \frac{1}{x} + \log x \cdot (-x \sin x + \cos x)$$

$$= \cos x - x \sin x \log x + \cos x \log x$$

$$\frac{du}{dx} = u [\cos x - x \sin x \log x + \cos x \log x]$$

$$= x^{x \cos x} [\cos x - x \sin x \log x + \cos x \log x] \quad \dots (2)$$

and,  $v = \frac{x^2 + 1}{x^2 - 1}$ , Taking log on both the sides

$\log v = \log(x^2 + 1) - \log(x^2 - 1)$ , therefore,

$$\frac{1}{v} \frac{dv}{dx} = \frac{1}{x^2 + 1} \cdot 2x - \frac{1}{x^2 - 1} \cdot 2x = \frac{2x(x^2 - 1) - 2x(x^2 + 1)}{(x^2 + 1)(x^2 - 1)} = \frac{-4x}{(x^2 + 1)(x^2 - 1)}$$

$$\frac{dv}{dx} = v \left[ \frac{-4x}{(x^2 + 1)(x^2 - 1)} \right] = \frac{x^2 + 1}{x^2 - 1} \left[ \frac{-4x}{(x^2 + 1)(x^2 - 1)} \right] = -\frac{4x}{(x^2 - 1)^2} \quad \dots (3)$$

Putting the value of  $\frac{du}{dx}$  from (2) and  $\frac{dv}{dx}$  from (3) in equation (1), we get

$$\frac{dy}{dx} = x^{x \cos x} [\cos x - x \sin x \log x + \cos x \log x] - \frac{4x}{(x^2 - 1)^2}$$

**Question 11:**

$$(x \cos x)^x + (x \sin x)^{\frac{1}{x}}$$

**Answer 11:**

Let  $u = (x \cos x)^x$  and  $v = (x \sin x)^{\frac{1}{x}}$ , therefore,  $y = u + v$

Differentiating with respect to  $x$ , we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots (1)$$

Here,  $u = (x \cos x)^x$ , Taking log on both the sides

$\log u = x \log(x \cos x)$ , therefore,

$$\frac{1}{u} \frac{du}{dx} = x \cdot \frac{d}{dx} \log(x \cos x) + \log(x \cos x) \cdot \frac{d}{dx} x$$

$$= x \cdot \frac{1}{(x \cos x)} (-x \sin x + \cos x) + \log(x \cos x) \cdot 1 = -x \tan x + 1 + \log(x \cos x)$$

$$\frac{du}{dx} = (x \cos x)^x [1 - x \tan x + \log(x \cos x)]$$

$$= (x \cos x)^x [1 - x \tan x + \log(x \cos x)] \quad \dots (2)$$

and,  $v = (x \sin x)^{\frac{1}{x}}$ , Taking log on both the sides

$\log v = \frac{1}{x} \log(x \sin x)$ , therefore,

$$\frac{1}{v} \frac{dv}{dx} = \frac{1}{x} \cdot \frac{d}{dx} \log(x \sin x) + \log(x \sin x) \cdot \frac{d}{dx} \frac{1}{x}$$

$$= \frac{1}{x} \cdot \frac{1}{x \sin x} (x \cos x + \sin x) + \log(x \sin x) \left(-\frac{1}{x^2}\right)$$

$$\frac{dv}{dx} = v \left[ \frac{x \cot x + 1 - \log(x \sin x)}{x^2} \right]$$

$$= (x \sin x)^{\frac{1}{x}} \left[ \frac{x \cot x + 1 - \log(x \sin x)}{x^2} \right] \quad \dots (3)$$

Putting the value of  $\frac{du}{dx}$  from (2) and  $\frac{dv}{dx}$  from (3) in equation (1), we get

$$\frac{dy}{dx} = (x \cos x)^x [1 - x \tan x + \log(x \cos x)] + (x \sin x)^{\frac{1}{x}} \left[ \frac{x \cot x + 1 - \log(x \sin x)}{x^2} \right]$$

Find  $\frac{dy}{dx}$  of the functions given in Exercises 12 to 15:

**Question 12:**

$$x^y + y^x = 1$$

**Answer 12:**

Let  $u = x^y$  and  $v = y^x$ , therefore,  $u + v = 1$

Differentiating with respect to  $x$ , we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \quad \dots (1)$$

Here,  $u = x^y$ , Taking log on both the sides,  $\log u = y \log x$ , therefore,

$$\frac{1}{u} \frac{du}{dx} = y \cdot \frac{d}{dx} \log x + \log x \cdot \frac{d}{dx} y = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\frac{du}{dx} = x^y \left[ \frac{y}{x} + \log x \cdot \frac{dy}{dx} \right] \quad \dots (2)$$

and,  $v = y^x$ , Taking log on both the sides

$\log v = x \log y$ , therefore,

$$\frac{1}{v} \frac{dv}{dx} = x \cdot \frac{d}{dx} \log y + \log y \cdot \frac{d}{dx} x = x \cdot \frac{1}{y} \frac{dy}{dx} + \log y \cdot 1$$

$$\frac{dv}{dx} = v \left[ \frac{x}{y} \frac{dy}{dx} + \log y \right] = y^x \left[ \frac{x}{y} \frac{dy}{dx} + \log y \right] \quad \dots (3)$$

Putting the value of  $\frac{du}{dx}$  from (2) and  $\frac{dv}{dx}$  from (3) in equation (1), we get

$$x^y \left[ \frac{y}{x} + \log x \cdot \frac{dy}{dx} \right] + y^x \left[ \frac{x}{y} \frac{dy}{dx} + \log y \right] = 0$$

$$\Rightarrow yx^{y-1} + x^y \log x \frac{dy}{dx} + xy^{x-1} \frac{dy}{dx} + y^x \log y = 0$$

$$\Rightarrow \frac{dy}{dx} (x^y \log x + xy^{x-1}) = -(y^x \log y + yx^{y-1})$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^x \log y + yx^{y-1}}{x^y \log x + xy^{x-1}}$$

### Question 13:

$$y^x = x^y$$

**Answer 13:**

$$y^x = x^y$$

Taking log on both the sides,  $x \log y = y \log x$ , therefore,

$$x \cdot \frac{d}{dx} \log y + \log y \cdot \frac{d}{dx} x = y \cdot \frac{d}{dx} \log x + \log x \cdot \frac{d}{dx} y$$

$$\Rightarrow x \cdot \frac{1}{y} \frac{dy}{dx} + \log y \cdot 1 = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} \left( \frac{x}{y} - \log x \right) = \frac{y}{x} - \log y$$

$$\Rightarrow \frac{dy}{dx} \left( \frac{x - y \log x}{y} \right) = \frac{y - x \log y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(y - x \log y)}{x(x - y \log x)}$$

### Question 14:

$$(\cos x)^y = (\cos y)^x$$

**Answer 14:**

$$(\cos x)^y = (\cos y)^x$$

Taking log on both the sides,  $y \log[\cos x] = x \log[\cos y]$ , therefore,

$$y \cdot \frac{d}{dx} \log \cos x + \log \cos x \cdot \frac{d}{dx} y = x \cdot \frac{d}{dx} \log \cos y + \log \cos y \cdot \frac{d}{dx} x$$

$$\Rightarrow y \left( \frac{-\sin x}{\cos x} \right) + \log \cos x \cdot \frac{dy}{dx} = x \cdot \left( \frac{-\sin y}{\cos y} \right) \frac{dy}{dx} + \log \cos y \cdot 1$$

$$\Rightarrow \frac{dy}{dx} (\log \cos x + x \tan y) = \log \cos y + y \tan x \quad \Rightarrow \frac{dy}{dx} = \frac{\log \cos y + y \tan x}{\log \cos x + x \tan y}$$

**Question 15:**

$$xy = e^{(x-y)}$$

**Answer 15:**

$$xy = e^{(x-y)}$$

Taking log on both the sides,

$$\log x + \log y = (x - y) \log e \Rightarrow \log x + \log y = (x - y), \text{ therefore,}$$

$$\frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} \left( \frac{1}{y} + 1 \right) = 1 - \frac{1}{x} \Rightarrow \frac{dy}{dx} \left( \frac{1+y}{y} \right) = \frac{x-1}{x} \Rightarrow \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$$

**Question 16:**

Find the derivative of the function given by  $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$  and hence find  $f'(1)$ .

**Answer 16:**

$$f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$$

Taking log on both the sides,

$$\log f(x) = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8), \text{ therefore,}$$

$$\frac{1}{f(x)} \cdot \frac{d}{dx} f(x) = \frac{1}{1+x} + \frac{1}{1+x^2} \cdot \frac{d}{dx} x^2 + \frac{1}{1+x^4} \cdot \frac{d}{dx} x^4 + \frac{1}{1+x^8} \cdot \frac{d}{dx} x^8$$

$$\Rightarrow \frac{1}{f(x)} \cdot f'(x) = \frac{1}{1+x} + \frac{1}{1+x^2} \cdot 2x + \frac{1}{1+x^4} \cdot 4x^3 + \frac{1}{1+x^8} \cdot 8x^7$$

$$\Rightarrow f'(x) = f(x) \left[ \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} \right]$$

$$\Rightarrow f'(x) = (1+x)(1+x^2)(1+x^4)(1+x^8) \left[ \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} \right]$$

$$\Rightarrow f'(1) = (1+1)(1+1)(1+1)(1+1) \left[ \frac{1}{1+1} + \frac{2}{1+1} + \frac{4}{1+1} + \frac{8}{1+1} \right]$$

$$\Rightarrow f'(1) = (2)(2)(2)(2) \left[ \frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2} \right] = 16 \left( \frac{15}{2} \right) = 120$$

**Question 17:**

Differentiate  $(x^2 - 5x + 8)(x^3 + 7x + 9)$  in three ways mentioned below:

(i) by using product rule

(ii) by expanding the product to obtain a single polynomial.

(iii) by logarithmic differentiation.

Do they all give the same answer?

**Answer 17:**

$$\text{Let } y = (x^2 - 5x + 8)(x^3 + 7x + 9)$$

(i) Differentiating using product rule

$$\frac{dy}{dx} = (x^2 - 5x + 8) \frac{d}{dx} (x^3 + 7x + 9) + (x^3 + 7x + 9) \frac{d}{dx} (x^2 - 5x + 8)$$

$$= (x^2 - 5x + 8)(3x^2 + 7) + (x^3 + 7x + 9)(2x - 5)$$

$$= (3x^4 + 7x^2 - 15x^3 - 35x + 24x^2 + 56) + 2x^4 - 5x^3 + 14x^2 - 35x + 18x - 45$$

$$= 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

**(ii)** Differentiating by expanding the product to obtain a single polynomial

$$y = (x^2 - 5x + 8)(x^3 + 7x + 9) = x^5 + 7x^3 + 9x^2 - 5x^4 - 35x^2 - 45x + 8x^3 + 56x + 72$$

$$= x^5 - 5x^4 + 15x^3 - 26x^2 + 11x + 72$$

$$\frac{dy}{dx} = \frac{d}{dx} x^5 - 5 \frac{d}{dx} x^4 + 15 \frac{d}{dx} x^3 - 26 \frac{d}{dx} x^2 + 11 \frac{d}{dx} x + \frac{d}{dx} 72 = 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

**(iii)** Logarithmic differentiation

$$y = (x^2 - 5x + 8)(x^3 + 7x + 9)$$

Taking log on both sides,  $\log y = \log(x^2 - 5x + 8) + \log(x^3 + 7x + 9)$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{(x^2 - 5x + 8)} \cdot \frac{d}{dx} (x^2 - 5x + 8) + \frac{1}{(x^3 + 7x + 9)} \cdot \frac{d}{dx} (x^3 + 7x + 9)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x^2 - 5x + 8} \cdot (2x - 5) + \frac{1}{x^3 + 7x + 9} \cdot (3x^2 + 7)$$

$$\frac{dy}{dx} = y \left[ \frac{(2x - 5)(x^3 + 7x + 9) + (3x^2 + 7)(x^2 - 5x + 8)}{(x^2 - 5x + 8)(x^3 + 7x + 9)} \right]$$

$$= y \left[ \frac{2x^4 + 14x^2 + 18x - 5x^3 - 35x - 45 + 3x^4 - 15x^3 + 24x^2 + 7x^2 - 35x + 56}{(x^2 - 5x + 8)(x^3 + 7x + 9)} \right]$$

$$\Rightarrow \frac{dy}{dx} = (x^2 - 5x + 8)(x^3 + 7x + 9) \left[ \frac{5x^5 - 20x^3 + 45x^2 - 52x + 11}{(x^2 - 5x + 8)(x^3 + 7x + 9)} \right]$$

$$\Rightarrow \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

Hence, all the three answers are same.

### Question 18:

If  $u, v$  and  $w$  are functions of  $x$ , then show that

$$\frac{d}{dx} (u \cdot v \cdot w) = \frac{du}{dx} \cdot v \cdot w + u \cdot \frac{dv}{dx} \cdot w + u \cdot v \cdot \frac{dw}{dx}$$

in two ways - first by repeated application of product rule, second by logarithmic differentiation.

#### Answer 18:

Let  $y = u \cdot v \cdot w = u \cdot (v \cdot w)$

Differentiation by repeated application of product rule

$$\frac{dy}{dx} = u \cdot \frac{d}{dx} (v \cdot w) + (v \cdot w) \cdot \frac{d}{dx} u = u \left[ v \frac{d}{dx} w + w \frac{d}{dx} v \right] + v \cdot w \cdot \frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = u \cdot v \cdot \frac{dw}{dx} + u \cdot w \cdot \frac{dv}{dx} + v \cdot w \cdot \frac{du}{dx}$$

Differentiation using logarithmic

Let  $y = u \cdot v \cdot w$

Taking log on both the sides,  $\log y = \log u + \log v + \log w$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx} + \frac{1}{w} \cdot \frac{dw}{dx} \Rightarrow \frac{dy}{dx} = y \left[ \frac{1}{u} \cdot \frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx} + \frac{1}{w} \cdot \frac{dw}{dx} \right]$$

$$\Rightarrow \frac{dy}{dx} = u \cdot v \cdot w \left[ \frac{1}{u} \cdot \frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx} + \frac{1}{w} \cdot \frac{dw}{dx} \right] \Rightarrow \frac{dy}{dx} = \frac{u \cdot v \cdot w}{u} \cdot \frac{du}{dx} + \frac{u \cdot v \cdot w}{v} \cdot \frac{dv}{dx} + \frac{u \cdot v \cdot w}{w} \cdot \frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = v \cdot w \cdot \frac{du}{dx} + u \cdot w \cdot \frac{dv}{dx} + u \cdot v \cdot \frac{dw}{dx}$$

# Mathematics

(Chapter - 5) (Continuity and Differentiability) (Exercise 5.6)  
(Class - XII)

If  $x$  and  $y$  are connected parametrically by the equations given in Exercises 1 to 10, without eliminating the parameter, Find  $\frac{dy}{dx}$ .

## Question 1:

$$x = 2at^2, y = at^4$$

### Answer 1:

$$\text{Here, } x = 2at^2, y = at^4$$

$$\text{Therefore, } \frac{dx}{dt} = 2a(2t) \text{ and } \frac{dy}{dt} = a(4t^3)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4at^3}{4at} = t^2$$

## Question 2:

$$x = a \cos \theta, y = b \cos \theta$$

### Answer 2:

$$\text{Here, } x = a \cos \theta, y = b \cos \theta, \text{ therefore, } \frac{dx}{d\theta} = a(-\sin \theta) \text{ and } \frac{dy}{d\theta} = b(-\sin \theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-b \sin \theta}{-a \sin \theta} = \frac{b}{a}$$

## Question 3:

$$x = \sin t, y = \cos 2t$$

### Answer 3:

$$\text{Here, } x = \sin t, y = \cos 2t, \text{ therefore, } \frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = -\sin 2t \cdot 2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2 \sin 2t}{\cos t} = -\frac{2(2 \sin t \cos t)}{\cos t} = -4 \sin t$$

## Question 4:

$$x = 4t, y = \frac{4}{t}$$

### Answer 4:

$$\text{Here, } x = 4t, y = \frac{4}{t}, \text{ therefore, } \frac{dx}{dt} = 4 \text{ and } \frac{dy}{dt} = -\frac{4}{t^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{4}{t^2}}{4} = -\frac{1}{t^2}$$

## Question 5:

$$x = \cos \theta - \cos 2\theta, y = \sin \theta - \sin 2\theta$$

### Answer 5:

$$\text{Here, } x = \cos \theta - \cos 2\theta, y = \sin \theta - \sin 2\theta$$

$$\text{Therefore, } \frac{dx}{d\theta} = -\sin \theta + 2 \sin 2\theta \text{ and } \frac{dy}{d\theta} = \cos \theta - 2 \cos 2\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos \theta - 2 \cos 2\theta}{-\sin \theta + 2 \sin 2\theta}$$

**Question 6:**

$$x = a(\theta - \sin \theta), y = a(1 + \cos \theta)$$

**Answer 6:**

$$\text{Here, } x = a(\theta - \sin \theta), y = a(1 + \cos \theta)$$

$$\text{Therefore, } \frac{dx}{d\theta} = a(1 - \cos \theta) \text{ and } \frac{dy}{d\theta} = a(0 - \sin \theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a \sin \theta}{a(1 - \cos \theta)} = -\frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} = -\cot \frac{\theta}{2}$$

**Question 7:**

$$x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

**Answer 7:**

$$\text{Here, } x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}, \text{ therefore, } \frac{dx}{dt} = \frac{\sin^3 t \frac{d}{dt} \sqrt{\cos 2t} - \sqrt{\cos 2t} \frac{d}{dt} \sin^3 t}{(\sqrt{\cos 2t})^2}$$

$$= \frac{\sin^3 t \cdot \frac{1}{2\sqrt{\cos 2t}} \cdot (-\sin 2t) \cdot 2 - \sqrt{\cos 2t} \cdot 3 \sin^2 t \cos t}{\cos 2t} = \frac{-\sin^3 t \cdot \sin 2t - 3 \cos 2t \cdot \sin^2 t \cos t}{\cos 2t \sqrt{\cos 2t}}$$

$$\text{and } \frac{dy}{dt} = \frac{\cos^3 t \frac{d}{dt} \sqrt{\cos 2t} - \sqrt{\cos 2t} \frac{d}{dt} \cos^3 t}{(\sqrt{\cos 2t})^2}$$

$$= \frac{\cos^3 t \cdot \frac{1}{2\sqrt{\cos 2t}} \cdot (-\sin 2t) \cdot 2 - \sqrt{\cos 2t} \cdot 3 \cos^2 t (-\sin t)}{\cos 2t} = \frac{-\cos^3 t \cdot \sin 2t + 3 \cos 2t \cdot \cos^2 t \sin t}{\cos 2t \sqrt{\cos 2t}}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\cos^3 t \cdot \sin 2t + 3 \cos 2t \cdot \cos^2 t \sin t}{-\sin^3 t \cdot \sin 2t - 3 \cos 2t \cdot \sin^2 t \cos t}$$

$$= \frac{-\cos^3 t \cdot (2 \sin t \cos t) + 3 \cos 2t \cdot \cos^2 t \sin t}{-\sin^3 t \cdot (2 \sin t \cos t) - 3 \cos 2t \cdot \sin^2 t \cos t} = \frac{\cos^2 t \sin t (-2 \cos^2 t + 3 \cos 2t)}{\sin^2 t \cos t (-2 \sin^2 t - 3 \cos 2t)}$$

$$= \frac{\cos t [-2 \cos^2 t + 3(2 \cos^2 t - 1)]}{\sin t [-2 \sin^2 t - 3(1 - 2 \sin^2 t)]} = \frac{\cos t [-2 \cos^2 t + 6 \cos^2 t - 3]}{\sin t [-2 \sin^2 t - 3 + 6 \sin^2 t]}$$

$$= \frac{\cos t [4 \cos^2 t - 3]}{\sin t [-3 + 4 \sin^2 t]} = -\frac{4 \cos^3 t - 3 \cos t}{3 \sin t - 4 \sin^3 t} = -\frac{\cos 3t}{\sin 3t} = -\cot 3t$$

**Question 8:**

$$x = a \left( \cos t + \log \tan \frac{t}{2} \right), y = a \sin t$$

**Answer 8:**

$$\text{Here, } x = a \left( \cos t + \log \tan \frac{t}{2} \right), y = a \sin t,$$

$$\text{Therefore, } \frac{dx}{dt} = a \left( -\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2} \right) = a \left( -\sin t + \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \cdot \frac{1}{\cos^2 \frac{t}{2}} \cdot \frac{1}{2} \right)$$

$$= a \left( -\sin t + \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}} \right) = a \left( -\sin t + \frac{1}{\sin t} \right) = a \left( \frac{-\sin^2 t + 1}{\sin t} \right) = a \left( \frac{\cos^2 t}{\sin t} \right)$$

$$\text{and } \frac{dy}{dt} = a \cos t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \cos t}{a \left( \frac{\cos^2 t}{\sin t} \right)} = \frac{\sin t}{\cos t} = \tan t$$

**Question 9:**

$$x = a \sec \theta, y = b \tan \theta$$

**Answer 9:**

$$\text{Here, } x = a \sec \theta, y = b \tan \theta$$

$$\text{Therefore, } \frac{dx}{d\theta} = a \sec \theta \tan \theta$$

$$\text{and } \frac{dy}{d\theta} = b \sec^2 \theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b \sec^2 \theta}{a \sec \theta \tan \theta} = \frac{b \sec \theta}{a \tan \theta} = \frac{b \left( \frac{1}{\cos \theta} \right)}{a \left( \frac{\sin \theta}{\cos \theta} \right)} = \frac{b}{a} \operatorname{cosec} \theta$$

**Question 10:**

$$x = a (\cos \theta + \theta \sin \theta), y = a (\sin \theta - \theta \cos \theta)$$

**Answer 10:**

$$\text{Here, } x = a (\cos \theta + \theta \sin \theta), y = a (\sin \theta - \theta \cos \theta)$$

$$\text{Therefore, } \frac{dx}{d\theta} = a [-\sin \theta + (\theta \cos \theta + \sin \theta)] = a \theta \cos \theta$$

$$\text{and } \frac{dy}{d\theta} = a [\cos \theta - (-\theta \sin \theta + \cos \theta)] = a \theta \sin \theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \theta \sin \theta}{a \theta \cos \theta} = \tan \theta$$

**Question 11:**

$$\text{If } x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}, \text{ show that } \frac{dy}{dx} = -\frac{y}{x}$$

**Answer 11:**

$$\text{Here, } x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$$

Therefore,

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{2\sqrt{a^{\sin^{-1} t}}} \cdot \frac{d}{dx} a^{\sin^{-1} t} = \frac{1}{2\sqrt{a^{\sin^{-1} t}}} \cdot a^{\sin^{-1} t} \cdot \log a \cdot \frac{1}{\sqrt{1-t^2}} \\ &= \frac{1}{2x} \cdot x^2 \cdot \log a \cdot \frac{1}{\sqrt{1-t^2}} = \frac{x \log a}{\sqrt{1-t^2}} \end{aligned}$$

and

$$\begin{aligned} \frac{dy}{dt} &= \frac{1}{2\sqrt{a^{\cos^{-1} t}}} \cdot \frac{d}{dx} a^{\cos^{-1} t} = \frac{1}{2\sqrt{a^{\cos^{-1} t}}} \cdot a^{\cos^{-1} t} \cdot \log a \cdot \frac{-1}{\sqrt{1-t^2}} \\ &= \frac{1}{2y} \cdot y^2 \cdot \log a \cdot \frac{1}{\sqrt{1-t^2}} = -\frac{y \log a}{\sqrt{1-t^2}} \end{aligned}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{y \log a}{\sqrt{1-t^2}}}{\frac{x \log a}{\sqrt{1-t^2}}} = -\frac{y}{x}$$

# Mathematics

(Chapter - 5) (Continuity and Differentiability) (Exercise 5.7)  
(Class - XII)

Find the second order derivatives of the functions given in Exercises 1 to 10.

## Question 1:

$$x^2 + 3x + 2$$

### Answer 1:

Let  $y = x^2 + 3x + 2$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 3x + 2) = 2x + 3$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(2x + 3) = 2$$

## Question 2:

$$x^{20}$$

### Answer 2:

Let  $y = x^{20}$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(x^{20}) = 20x^{19}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(20x^{19}) = 380x^{18}$$

## Question 3:

$$x \cdot \cos x$$

### Answer 3:

Let  $y = x \cdot \cos x$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(x \cdot \cos x) = x \cdot \frac{d}{dx} \cos x + \cos x \cdot \frac{d}{dx} x = -x \sin x + \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(-x \sin x + \cos x) = -\left(x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} x\right) - \sin x$$

$$= -x \cos x - \sin x - \sin x = -(x \cos x + 2 \sin x)$$

## Question 4:

$$\log x$$

### Answer 4:

Let  $y = \log x$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

## Question 5:

$$x^3 \log x$$

### Answer 5:

Let  $y = x^3 \log x$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 \log x) = x^3 \cdot \frac{d}{dx} \log x + \log x \cdot \frac{d}{dx} x^3 = x^3 \cdot \frac{1}{x} + \log x \cdot 3x^2 = x^2 + 3x^2 \log x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(x^2 + 3x^2 \log x) = 2x + 3\left(x^2 \frac{d}{dx} \log x + \log x \frac{d}{dx} x^2\right)$$

$$= 2x + 3\left(x^2 \cdot \frac{1}{x} + \log x \cdot 2x\right) = 2x + 3x + 6x \log x = 5x + 6x \log x = x(5 + 6 \log x)$$

**Question 6:**

$$e^x \sin 5x$$

**Answer 6:**

Let  $y = e^x \sin 5x$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \sin 5x) = e^x \cdot \frac{d}{dx} \sin 5x + \sin 5x \cdot \frac{d}{dx} e^x$$

$$= e^x \cdot \cos 5x \cdot 5 + \sin 5x \cdot e^x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(5e^x \cos 5x + e^x \sin 5x)$$

$$= 5 \left( e^x \cdot \frac{d}{dx} \cos 5x + \cos 5x \cdot \frac{d}{dx} e^x \right) + \left( e^x \cdot \frac{d}{dx} \sin 5x + \sin 5x \cdot \frac{d}{dx} e^x \right)$$

$$= 5[e^x \cdot (-\sin 5x) \cdot 5 + \cos 5x \cdot e^x] + [e^x \cdot \cos 5x \cdot 5 + \sin 5x \cdot e^x]$$

$$= e^x(-25 \sin 5x + 5 \cos 5x + 5 \cos 5x + \sin 5x) = e^x(10 \cos 5x - 24 \sin 5x)$$

**Question 7:**

$$e^{6x} \cos 3x$$

**Answer 7:**

Let  $y = e^{6x} \cos 3x$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(e^{6x} \cos 3x) = e^{6x} \cdot \frac{d}{dx} \cos 3x + \cos 3x \cdot \frac{d}{dx} e^{6x}$$

$$= e^{6x} \cdot (-\sin 3x) \cdot 3 + \cos 3x \cdot e^{6x} \cdot 6$$

$$= 3e^{6x}(-\sin 3x + 2 \cos 3x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}[3e^{6x}(-\sin 3x + 2 \cos 3x)]$$

$$= 3e^{6x} \cdot \frac{d}{dx}(-\sin 3x + 2 \cos 3x) + (-\sin 3x + 2 \cos 3x) \cdot \frac{d}{dx} 3e^{6x}$$

$$= 3e^{6x} \cdot (-3 \cos 3x - 6 \sin 3x) + (-\sin 3x + 2 \cos 3x) \cdot 18e^{6x}$$

$$= e^{6x}(-9 \cos 3x - 18 \sin 3x - 18 \sin 3x + 36 \cos 3x)$$

$$= e^{6x}(27 \cos 3x - 36 \sin 3x)$$

$$= 9e^{6x}(3 \cos 3x - 4 \sin 3x)$$

**Question 8:**

$$\tan^{-1} x$$

**Answer 8:**

Let  $y = \tan^{-1} x$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{1}{1+x^2} \right) = \frac{(1+x^2) \frac{d}{dx} 1 - 1 \cdot \frac{d}{dx} (1+x^2)}{(1+x^2)^2}$$

$$= \frac{0 - 2x}{(1+x^2)^2} = -\frac{2x}{(1+x^2)^2}$$

**Question 9:**

$\log(\log x)$

**Answer 9:**

Let  $y = \log(\log x)$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(\log(\log x)) = \frac{1}{\log x} \cdot \frac{1}{x} = \frac{1}{x \log x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{1}{x \log x} \right) = \frac{(x \log x) \frac{d}{dx} 1 - 1 \cdot \frac{d}{dx} (x \log x)}{(x \log x)^2}$$

$$= \frac{0 - \left( x \cdot \frac{1}{x} + \log x \right)}{(x \log x)^2} = -\frac{1 + \log x}{(x \log x)^2}$$

**Question 10:**

$\sin(\log x)$

**Answer 10:**

Let  $y = \sin(\log x)$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(\sin(\log x)) = \cos(\log x) \cdot \frac{1}{x} = \frac{\cos(\log x)}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{\cos(\log x)}{x} \right] = \frac{x \frac{d}{dx} \cos(\log x) - \cos(\log x) \cdot \frac{d}{dx} x}{(x)^2}$$

$$= \frac{x \left\{ -\sin(\log x) \cdot \frac{1}{x} \right\} - \cos(\log x) \cdot 1}{(x)^2} = \frac{-\sin(\log x) - \cos(\log x)}{(x)^2}$$

**Question 11:**

If  $y = 5 \cos x - 3 \sin x$ , prove that  $\frac{d^2y}{dx^2} + y = 0$

**Answer 11:**

Given that:  $y = 5 \cos x - 3 \sin x$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx}(5 \cos x - 3 \sin x) = -5 \sin x - 3 \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(-5 \sin x - 3 \cos x) = -5 \cos x + 3 \sin x = -(5 \cos x - 3 \sin x) = -y$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$

**Question 12:**

If  $y = \cos^{-1} x$ , find  $\frac{d^2y}{dx^2}$  in terms of  $y$  alone.

**Answer 12:**

Given that:  $y = \cos^{-1} x$

$\Rightarrow \cos y = x$ , therefore,

$$-\sin y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y} = -\operatorname{cosec} y$$

$$\Rightarrow \frac{d^2y}{dx^2} = -(-\operatorname{cosec} y \cot y) \cdot \frac{dy}{dx} = (\operatorname{cosec} y \cot y) \cdot (-\operatorname{cosec} y) = -\operatorname{cosec}^2 y \cot y$$

**Question 13:**

If  $y = 3 \cos(\log x) + 4 \sin(\log x)$ , show that  $x^2 y_2 + x y_1 + y = 0$

**Answer 13:**

Given that:  $y = 3 \cos(\log x) + 4 \sin(\log x)$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx} (3 \cos(\log x) + 4 \sin(\log x)) = -3 \sin(\log x) \cdot \frac{1}{x} + 4 \cos(\log x) \cdot \frac{1}{x}$$

$$\Rightarrow x \frac{dy}{dx} = -3 \sin(\log x) + 4 \cos(\log x)$$

$$\Rightarrow x \frac{d^2 y}{dx^2} + \frac{dy}{dx} \cdot \frac{d}{dx} x = \frac{d}{dx} [-3 \sin(\log x) + 4 \cos(\log x)]$$

$$= -3 \cos(\log x) \cdot \frac{1}{x} - 4 \sin(\log x) \cdot \frac{1}{x} = -\frac{1}{x} [3 \cos(\log x) + 4 \sin(\log x)] = -\frac{1}{x} \cdot y$$

$$\Rightarrow x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x} y \quad \Rightarrow x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = -y \quad \Rightarrow x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$$

$$\Rightarrow x^2 y_2 + x y_1 + y = 0$$

**Question 14:**

If  $y = Ae^{mx} + Be^{nx}$ , show that  $\frac{d^2 y}{dx^2} - (m+n) \frac{dy}{dx} + mny = 0$

**Answer 14:**

Given that:  $y = Ae^{mx} + Be^{nx}$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx} (Ae^{mx} + Be^{nx}) = mAe^{mx} + nBe^{nx}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{d}{dx} (mAe^{mx} + nBe^{nx}) = m^2 Ae^{mx} + n^2 Be^{nx}$$

Putting the value of  $\frac{d^2 y}{dx^2}$  and  $\frac{dy}{dx}$  in  $\frac{d^2 y}{dx^2} - (m+n) \frac{dy}{dx} + mny$ , we get

$$\begin{aligned} \text{LHS} &= (m^2 Ae^{mx} + n^2 Be^{nx}) - (m+n)(mAe^{mx} + nBe^{nx}) + mny \\ &= m^2 Ae^{mx} + n^2 Be^{nx} - (m^2 Ae^{mx} + mnBe^{nx} + mnAe^{mx} + n^2 Be^{nx}) + mny \\ &= -(mnAe^{mx} + mnBe^{nx}) + mny \\ &= -mn(Ae^{mx} + Be^{nx}) + mny \\ &= -mny + mny = 0 = \text{RHS} \end{aligned}$$

**Question 15:**

If  $y = 500e^{7x} + 600e^{-7x}$ , Show that  $\frac{d^2 y}{dx^2} = 49y$ .

**Answer 15:**

Given that:  $y = 500e^{7x} + 600e^{-7x}$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx} (500e^{7x} + 600e^{-7x}) = 500e^{7x} \cdot 7 + 600e^{-7x} \cdot (-7) = 7(500e^{7x} - 600e^{-7x})$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{d}{dx} 7(500e^{7x} - 600e^{-7x}) = 7[500e^{7x} \cdot 7 + 600e^{-7x} \cdot (-7)]$$

$$= 49(500e^{7x} - 600e^{-7x}) = 49y$$

$$\Rightarrow \frac{d^2 y}{dx^2} = 49y$$

**Question 16:**

If  $e^y(x+1) = 1$ , show that  $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$ .

**Answer 16:**

Given that:  $e^y(x+1) = 1$ , therefore,

$$e^y \frac{dy}{dx}(x+1) + (x+1) \frac{d}{dx} e^y = \frac{d}{dx} 1$$

$$\Rightarrow e^y + (x+1)e^y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x+1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \left( -\frac{1}{x+1} \right)$$

$$= - \left[ \frac{(x+1) \cdot \frac{d}{dx} 1 - 1 \cdot \frac{d}{dx} (x+1)}{(x+1)^2} \right] = - \left[ \frac{0 - 1}{(x+1)^2} \right] = \frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left( -\frac{1}{x+1} \right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left( \frac{dy}{dx} \right)^2$$

**Question 17:**

If  $y = (\tan^{-1} x)^2$ , show that  $(x^2 + 1)^2 y_2 + 2x(x^2 + 1)y_1 = 2$ .

**Answer 17:**

Given that:  $y = (\tan^{-1} x)^2$ , therefore,

$$\frac{dy}{dx} = \frac{d}{dx} [(\tan^{-1} x)^2] = 2 \tan^{-1} x \cdot \frac{1}{1+x^2} = \frac{2 \tan^{-1} x}{1+x^2}$$

$$\Rightarrow (1+x^2) \frac{dy}{dx} = 2 \tan^{-1} x$$

$$\Rightarrow (1+x^2) \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{d}{dx} (1+x^2) = \frac{d}{dx} (2 \tan^{-1} x)$$

$$\Rightarrow (1+x^2) \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 2x = \frac{2}{1+x^2}$$

$$\Rightarrow (1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2) \frac{dy}{dx} = 2$$

$$\Rightarrow (x^2 + 1)^2 y_2 + 2x(x^2 + 1)y_1 = 2$$

# Mathematics

(Chapter - 5) (Continuity and Differentiability) (Miscellaneous Exercise)  
(Class - XII)

Differentiate w.r.t.  $x$  the function in Exercises 1 to 11.

## Question 1:

$$(3x^2 - 9x + 5)^9$$

### Answer 1:

Let  $y = (3x^2 - 9x + 5)^9$ , therefore,

$$\begin{aligned}\frac{dy}{dx} &= 9(3x^2 - 9x + 5)^8 \cdot \frac{d}{dx}(3x^2 - 9x + 5) = 9(3x^2 - 9x + 5)^8 \cdot (6x - 9) \\ &= 27(3x^2 - 9x + 5)^8 \cdot (2x - 3)\end{aligned}$$

## Question 2:

$$\sin^3 x + \cos^6 x$$

### Answer 2:

Let  $y = \sin^3 x + \cos^6 x$ , therefore,

$$\begin{aligned}\frac{dy}{dx} &= 3 \sin^2 x \cdot \frac{d}{dx} \sin x + 6 \cos^5 x \cdot \frac{d}{dx} \cos x = 3 \sin^2 x \cdot \cos x + 6 \cos^5 x \cdot (-\sin x) \\ &= 3 \sin x \cos x (\sin x - 2 \cos^4 x)\end{aligned}$$

## Question 3:

$$(5x)^{3 \cos 2x}$$

### Answer 3:

Let  $y = (5x)^{3 \cos 2x}$ , taking log on both sides

$$\log y = \log(5x)^{3 \cos 2x} = 3 \cos 2x \cdot \log 5x$$

Therefore,

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= 3 \cos 2x \cdot \frac{d}{dx} \log 5x + \log 5x \cdot \frac{d}{dx} 3 \cos 2x \\ \Rightarrow \frac{dy}{dx} &= y \left[ 3 \cos 2x \cdot \frac{1}{5x} \cdot 5 + \log 5x \cdot 3(-\sin 2x) \cdot 2 \right] \\ \Rightarrow \frac{dy}{dx} &= 3(5x)^{3 \cos 2x} \left[ \frac{\cos 2x - 2 \sin 2x \log 5x}{x} \right]\end{aligned}$$

## Question 4:

$$\sin^{-1}(x\sqrt{x}), 0 \leq x \leq 1$$

### Answer 4:

Let  $y = \sin^{-1}(x\sqrt{x})$ , therefore,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{1 - (x\sqrt{x})^2}} \cdot \frac{d}{dx}(x\sqrt{x}) = \frac{1}{\sqrt{1 - x^3}} \cdot \left[ x \frac{d}{dx} \sqrt{x} + \sqrt{x} \cdot \frac{d}{dx} x \right] \\ &= \frac{1}{\sqrt{1 - x^3}} \cdot \left[ x \frac{1}{2\sqrt{x}} + \sqrt{x} \cdot 1 \right] \\ &= \frac{1}{\sqrt{1 - x^3}} \cdot \left[ \frac{x + 2x}{2\sqrt{x}} \right] \\ &= \frac{3x}{2\sqrt{x}\sqrt{1 - x^3}} = \frac{3}{2} \sqrt{\frac{x}{1 - x^3}}\end{aligned}$$

**Question 5:**

$$\frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2.$$

**Answer 5:**

Let  $y = \frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}$ , therefore,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos^{-1} \frac{x}{2} \cdot \frac{d}{dx} \sqrt{2x+7} - \sqrt{2x+7} \frac{d}{dx} \cos^{-1} \frac{x}{2}}{(\sqrt{2x+7})^2} \\ &= \frac{\left[ \cos^{-1} \frac{x}{2} \cdot \frac{1}{2\sqrt{2x+7}} \cdot 2 \right] - \sqrt{2x+7} \frac{-1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}} \cdot \frac{1}{2}}{2x+7} \\ &= \frac{\cos^{-1} \frac{x}{2} \cdot \frac{1}{\sqrt{2x+7}} + \sqrt{2x+7} \frac{1}{\sqrt{4 - (x)^2}}}{2x+7} = \frac{\cos^{-1} \frac{x}{2} \cdot \sqrt{4 - x^2} + 2x + 7}{(2x+7)\sqrt{2x+7}\sqrt{4 - x^2}} \end{aligned}$$

**Question 6:**

$$\cot^{-1} \left[ \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right], 0 < x < \frac{\pi}{2}$$

**Answer 6:**

Let  $y = \cot^{-1} \left[ \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right]$ , therefore,

$$\begin{aligned} y &= \cot^{-1} \left[ \frac{\sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}} + \sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2\sin \frac{x}{2} \cos \frac{x}{2}}}{\sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}} - \sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2\sin \frac{x}{2} \cos \frac{x}{2}}} \right] \\ &= \cot^{-1} \left[ \frac{\sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)^2} + \sqrt{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)^2}}{\sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)^2} - \sqrt{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)^2}} \right] \\ &= \cot^{-1} \left[ \frac{\cos \frac{x}{2} + \sin \frac{x}{2} + \cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2} - \cos \frac{x}{2} + \sin \frac{x}{2}} \right] = \cot^{-1} \left[ \frac{2\cos \frac{x}{2}}{2\sin \frac{x}{2}} \right] = \cot^{-1} \left[ \cot \frac{x}{2} \right] = \frac{x}{2} \end{aligned}$$

Therefore,  $\frac{dy}{dx} = \frac{1}{2}$

**Question 7:**

$$(\log x)^{\log x}, x > 1$$

**Answer 7:**

Let  $y = (\log x)^{\log x}$ , taking log on both sides

$$\log y = \log(\log x)^{\log x} = \log x \cdot \log(\log x)$$

Therefore,

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \log x \cdot \frac{d}{dx} \log(\log x) + \log(\log x) \cdot \frac{d}{dx} \log x \\ \Rightarrow \frac{dy}{dx} &= y \left[ \log x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \log(\log x) \cdot \frac{1}{x} \right] \Rightarrow \frac{dy}{dx} = (\log x)^{\log x} \left[ \frac{1 + \log(\log x)}{x} \right] \end{aligned}$$

**Question 8:**

$\cos(a \cos x + b \sin x)$ , for some constant  $a$  and  $b$ .

**Answer 8:**

Let  $y = \cos(a \cos x + b \sin x)$ , therefore,

$$\frac{dy}{dx} = -\sin(a \cos x + b \sin x) \cdot \frac{d}{dx}(a \cos x + b \sin x)$$

$$= -\sin(a \cos x + b \sin x) (-a \sin x + b \cos x)$$

$$= \sin(a \cos x + b \sin x) (a \sin x - b \cos x)$$

**Question 9:**

$(\sin x - \cos x)^{(\sin x - \cos x)}$ ,  $\frac{\pi}{4} < x < \frac{3\pi}{4}$

**Answer 9:**

Let  $y = (\sin x - \cos x)^{(\sin x - \cos x)}$ , taking log on both sides

$$\log y = \log(\sin x - \cos x)^{(\sin x - \cos x)} = (\sin x - \cos x) \cdot \log(\sin x - \cos x)$$

Therefore,

$$\frac{1}{y} \frac{dy}{dx} = (\sin x - \cos x) \cdot \frac{d}{dx} \log(\sin x - \cos x) + \log(\sin x - \cos x) \cdot \frac{d}{dx} (\sin x - \cos x)$$

$$\Rightarrow \frac{dy}{dx} = y \left[ (\sin x - \cos x) \cdot \frac{(\cos x + \sin x)}{(\sin x - \cos x)} + \log(\sin x - \cos x) (\cos x + \sin x) \right]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)} (\cos x + \sin x) [1 + \log(\cos x - \sin x)]$$

**Question 10:**

$x^x + x^a + a^x + a^a$ , for some fixed  $a > 0$  and  $x > 0$ .

**Answer 10:**

Let  $u = x^x$  and  $y = u + x^a + a^x + a^a$  therefore,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{d}{dx} x^a + \frac{d}{dx} a^x + \frac{d}{dx} a^a$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + ax^{a-1} + a^x \log a + 0 \quad \dots (1)$$

Here,  $u = x^x$ , taking log on both sides

$$\log u = \log x^x = x \cdot \log x$$

Therefore,

$$\frac{1}{u} \frac{du}{dx} = x \cdot \frac{d}{dx} \log x + \log x \cdot \frac{d}{dx} x = x \cdot \frac{1}{x} + \log x \cdot 1 \Rightarrow \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x)$$

Putting the value of  $\frac{du}{dx}$  in equation (1), we get

$$\frac{dy}{dx} = x^x(1 + \log x) + ax^{a-1} + a^x \log a$$

**Question 11:**

$x^{x^2-3} + (x-3)^{x^2}$ , for  $x > 3$ .

**Answer 11:**

Let  $u = x^{x^2-3}$  and  $v = (x-3)^{x^2}$  therefore,  $y = u + v$

Differentiating with respect to  $x$ , we have

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots (1)$$

Here,  $u = x^{x^2-3}$ , taking log on both sides  
 $\log u = (x^2 - 3) \log x$ , therefore,

$$\frac{1}{u} \frac{du}{dx} = (x^2 - 3) \cdot \frac{d}{dx} \log x + \log x \cdot \frac{d}{dx} (x^2 - 3) = (x^2 - 3) \cdot \frac{1}{x} + \log x \cdot 2x$$

$$\frac{du}{dx} = u \left[ \frac{x^2 - 3 + 2x^2 \log x}{x} \right]$$

$$\frac{du}{dx} = x^{x^2-3} \left[ \frac{x^2 - 3 + 2x^2 \log x}{x} \right] = x^{x^2-4} (x^2 - 3 + 2x^2 \log x) \quad \dots (2)$$

and,  $v = (x - 3)^{x^2}$ , taking log on both sides  
 $\log v = x^2 \log(x - 3)$ , therefore,

$$\frac{1}{v} \frac{dv}{dx} = x^2 \cdot \frac{d}{dx} \log(x - 3) + \log(x - 3) \cdot \frac{d}{dx} x^2 = x^2 \cdot \frac{1}{x - 3} + \log(x - 3) \cdot 2x = \frac{x^2}{x - 3} + 2x \cdot \log(x - 3)$$

$$\frac{dv}{dx} = v \left[ \frac{x^2}{x - 3} + 2x \cdot \log(x - 3) \right] = (x - 3)^{x^2} \left[ \frac{x^2}{x - 3} + 2x \cdot \log(x - 3) \right] \quad \dots (3)$$

Putting the value of  $\frac{du}{dx}$  from (2) and value of  $\frac{dv}{dx}$  from (3) in equation (1), we have

$$\frac{dy}{dx} = x^{x^2-4} (x^2 - 3 + 2x^2 \log x) + (x - 3)^{x^2} \left[ \frac{x^2}{x - 3} + 2x \cdot \log(x - 3) \right]$$

### Question 12:

Find  $\frac{dy}{dx}$ , if  $y = 12(1 - \cos t)$ ,  $x = 10(t - \sin t)$ ,  $-\frac{\pi}{2} < t < \frac{\pi}{2}$ .

#### Answer 12:

Here,  $x = 10(t - \sin t)$ ,  $y = 12(1 - \cos t)$

Therefore,  $\frac{dx}{dt} = 10(1 - \cos t)$  and  $\frac{dy}{dt} = 12(0 + \sin t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12 \sin t}{10(1 - \cos t)} = \frac{6 \left( 2 \sin \frac{t}{2} \cos \frac{t}{2} \right)}{5 \left( 2 \sin^2 \frac{t}{2} \right)} = \frac{6}{5} \cot \frac{t}{2}$$

### Question 13:

Find  $\frac{dy}{dx}$ , if  $y = \sin^{-1} x + \sin^{-1} \sqrt{1 - x^2}$ ,  $0 < x < 1$ .

#### Answer 13:

Here,  $y = \sin^{-1} x + \sin^{-1} \sqrt{1 - x^2}$ , therefore

$$\frac{dy}{dx} = \frac{d}{dx} \sin^{-1} x + \frac{d}{dx} \sin^{-1} \sqrt{1 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - (\sqrt{1 - x^2})^2}} \frac{d}{dx} \sqrt{1 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} + \frac{1}{x} \cdot \frac{1}{2\sqrt{1 - x^2}} \frac{d}{dx} (1 - x^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} + \frac{1}{x} \cdot \frac{1}{2\sqrt{1 - x^2}} (-2x) = \frac{1}{\sqrt{1 - x^2}} - \frac{1}{\sqrt{1 - x^2}} = 0$$

**Question 14:**

If  $x\sqrt{1+y} + y\sqrt{1+x} = 0$ , for  $-1 < x < 1$ , Prove that  $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

**Answer 14:**

$$\text{Given that: } x\sqrt{1+y} + y\sqrt{1+x} = 0 \quad \Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$$

Squaring both sides

$$x^2(1+y) = y^2(1+x) \quad \Rightarrow x^2 + x^2y = y^2 + y^2x$$

$$\Rightarrow x^2 - y^2 + x^2y - y^2x = 0$$

$$\Rightarrow (x+y)(x-y) + xy(x-y) = 0 \quad \Rightarrow (x-y)(x+y+xy) = 0$$

$$\Rightarrow (x+y+xy) = 0 \quad [\because x \neq y \Rightarrow x-y \neq 0]$$

$$\Rightarrow y(1+x) = -x \Rightarrow y = -\frac{x}{1+x}$$

Therefore,

$$\frac{dy}{dx} = - \left[ \frac{(1+x) \frac{d}{dx} x - x \frac{d}{dx} (1+x)}{(1+x)^2} \right] = - \frac{1+x-x}{(1+x)^2} = - \frac{1}{(1+x)^2}$$

**Question 15:**

If  $(x-a)^2 + (y-b)^2 = c^2$  for some  $c > 0$ , prove that

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

is a constant independent of  $a$  and  $b$ .

**Answer 15:**

Given that:  $(x-a)^2 + (y-b)^2 = c^2$ . Differentiating with respect to  $x$ , we have

$$\frac{d}{dx}(x-a)^2 + \frac{d}{dx}(y-b)^2 = \frac{d}{dx}c^2 \quad \Rightarrow 2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x-a}{y-b}$$

Differentiating again, we have

$$\frac{d^2y}{dx^2} = - \frac{(y-b) \frac{d}{dx}(x-a) - (x-a) \frac{d}{dx}(y-b)}{(y-b)^2} = - \frac{(y-b)1 - (x-a)\frac{dy}{dx}}{(y-b)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = - \frac{(y-b)1 - (x-a)\left(-\frac{x-a}{y-b}\right)}{(y-b)^2} = - \frac{(y-b)^2 + (x-a)^2}{(y-b)^3} = - \frac{c^2}{(y-b)^3}$$

Putting the values in  $\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ , we have

$$\begin{aligned} \frac{\left[1 + \left(-\frac{x-a}{y-b}\right)^2\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}} &= \frac{\left[1 + \frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}} = \frac{\left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}} = \frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}} \\ &= \frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}} = -\frac{c^3}{c^2} = -c, \text{ which is a constant independent of } a \text{ and } b. \end{aligned}$$

**Question 16:**

If  $\cos y = x \cos(a + y)$ , with  $\cos a \neq \pm 1$ , prove that  $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

**Answer 16:**

Given that:  $\cos y = x \cos(a + y) \Rightarrow x = \frac{\cos y}{\cos(a+y)}$ , therefore,

Differentiating with respect to  $y$ , we have

$$\frac{dx}{dy} = \frac{\cos(a+y) \frac{d}{dy} \cos y - \cos y \frac{d}{dy} \cos(a+y)}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\cos(a+y)(-\sin y) - \cos y(-\sin(a+y))}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{-\sin y \cos(a+y) + \cos y \sin(a+y)}{\cos^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\cos^2(a+y)} = \frac{\sin a}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

**Question 17:**

If  $x = a(\cos t + t \sin t)$  and  $y = a(\sin t - t \cos t)$ , find  $\frac{d^2y}{dx^2}$ .

**Answer 17:**

Here,  $x = a(\cos t + t \sin t)$ ,  $y = a(\sin t - t \cos t)$

Therefore,

$$\frac{dx}{dt} = a[-\sin t + (t \cos t + \sin t)] = at \cos t$$

and

$$\frac{dy}{dt} = a[(\cos t - (-t \sin t + \cos t))] = at \sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at \sin t}{at \cos t} = \tan t$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx} = \sec^2 t \cdot \frac{1}{at \cos t} = \frac{\sec^3 t}{at}$$

**Question 18:**

If  $f(x) = |x|^3$ , show that  $f''(x)$  exists for all real  $x$  and find it.

**Answer 18:**

Rewriting the function  $f(x) = |x|^3$  in the following form:

$$f(x) = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}$$

$$\text{If } x \geq 0, f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x$$

$$\text{If } x < 0, f(x) = -x^3 \Rightarrow f'(x) = -3x^2 \Rightarrow f''(x) = -6x$$

Hence,  $f''(x)$  exists for all real  $x$  and it can be represented as follows:

$$f''(x) = \begin{cases} 6x & \text{if } x \geq 0 \\ -6x & \text{if } x < 0 \end{cases}$$

**Question 19:**

Using the fact that  $\sin(A + B) = \sin A \cos B + \cos A \sin B$  and the differentiation, obtain the sum formula for cosines.

**Answer 19:**

Given that:  $\sin(A + B) = \sin A \cos B + \cos A \sin B$ ,

Differentiating with respect to  $x$ , we have

$$\begin{aligned} \frac{d}{dx} \sin(A + B) &= \left( \sin A \frac{d}{dx} \cos B + \cos B \frac{d}{dx} \sin A \right) + \left( \cos A \frac{d}{dx} \sin B + \sin B \frac{d}{dx} \cos A \right) \\ &\Rightarrow \cos(A + B) \cdot \left( \frac{dA}{dx} + \frac{dB}{dx} \right) \\ &= \left( \sin A (-\sin B) \frac{dB}{dx} + \cos B \cos A \frac{dA}{dx} \right) + \left( \cos A \cos B \frac{dB}{dx} + \sin B (-\sin A) \frac{dA}{dx} \right) \\ &\Rightarrow \cos(A + B) \cdot \left( \frac{dA}{dx} + \frac{dB}{dx} \right) \\ &= (\cos A \cos B - \sin A \sin B) \frac{dB}{dx} + (\cos A \cos B - \sin A \sin B) \frac{dA}{dx} \\ &\Rightarrow \cos(A + B) \cdot \left( \frac{dA}{dx} + \frac{dB}{dx} \right) = (\cos A \cos B - \sin A \sin B) \left( \frac{dA}{dx} + \frac{dB}{dx} \right) \\ &\Rightarrow \cos(A + B) = \cos A \cos B - \sin A \sin B \end{aligned}$$

**Question 20:**

Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

**Answer 20:**

Function  $f(x) = |x - 1| + |x - 3|$  is continuous for all real points but not differentiable at two points ( $x = 1$  and  $x = 3$ ).

**Question 21:**

If  $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$  prove that  $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

**Answer 21:**

Given that:  $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$ , therefore,

$$\begin{aligned} \frac{dy}{dx} &= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \frac{dl}{dx} & \frac{dm}{dx} & \frac{dn}{dx} \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ \frac{da}{dx} & \frac{db}{dx} & \frac{dc}{dx} \end{vmatrix} \\ &\Rightarrow \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix} \\ &\Rightarrow \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} + 0 + 0 \Rightarrow \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix} \end{aligned}$$

**Question 22:**

If  $y = e^{a \cos^{-1} x}$ ,  $-1 \leq x \leq 1$ , show that  $(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$

**Answer 22:**

Given that:  $y = e^{a \cos^{-1} x}$ , therefore,

Differentiating with respect to  $x$ , we have

$$\frac{dy}{dx} = \frac{d}{dx} e^{a \cos^{-1} x} = e^{a \cos^{-1} x} \frac{d}{dx} a \cos^{-1} x$$

$$\Rightarrow \frac{dy}{dx} = e^{a \cos^{-1} x} a \cdot \frac{-1}{\sqrt{1-x^2}} = -\frac{ay}{\sqrt{1-x^2}}$$

Squaring both the sides, we have

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2 y^2}{1-x^2}$$

$$\Rightarrow (1-x^2) \left(\frac{dy}{dx}\right)^2 = a^2 y^2$$

Differentiating again with respect to  $x$ , we have

$$(1-x^2) \cdot 2 \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \frac{d}{dx} (1-x^2) = a^2 2y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} \left[ 2(1-x^2) \frac{d^2 y}{dx^2} + \frac{dy}{dx} (-2x) \right] = 2a^2 y \frac{dy}{dx}$$

$$\Rightarrow 2 \frac{dy}{dx} \left[ (1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} \right] = 2a^2 y \frac{dy}{dx}$$

$$\Rightarrow (1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = a^2 y$$

$$\Rightarrow (1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$$

