Mathematics

(Chapter – 11) (Three Dimensional Geometry) (Exercise 11.1)
(Class – XII)

Question 1:

If a line makes angles 90°, 135°, 45° with the x, y and z-axes respectively, find its direction cosines.

Answer 1:

Let direction cosines of the line be l, m, and n.

$l = \cos 90^0 = 0$

$m = \cos 135^0 = -\frac{1}{\sqrt{2}}$

$n = \cos 45^0 = \frac{1}{\sqrt{2}}$

Therefore, the direction cosines of the line are

$0, -\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$

Question 2:

Find the direction cosines of a line which makes equal angles with the coordinate axes.

Answer 2:

Let the direction cosines of the line make an angle α with each of the coordinate axes.

$l = \cos\alpha$, $m = \cos\alpha$ and $n = \cos\alpha$

$cos^2\alpha + cos^2\alpha + cos^2\alpha = 1$

$\Rightarrow 3cos^2\alpha = 1$

$\Rightarrow cos^2\alpha = \frac{1}{\sqrt{3}}$

$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$

Thus, the direction cosines of the line, which is equally inclined to the coordinate axes, are

$\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$ and $\pm\frac{1}{\sqrt{3}}$

Question 3:

If a line has the direction ratios −18, 12, −4, then what are its direction cosines?

Answer 3:

If a line has direction ratios of −18, 12, and −4, then its direction cosines are

$$\frac{-18}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}, \frac{12}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}, \frac{-4}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}$$

$= i.e. \frac{-18}{22}, \frac{12}{22}, \frac{-4}{22}$

$= \frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$

Thus, the direction cosines are

$\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$

Question 4:

Show that the points $(2, 3, 4)$, $(-1, -2, 1)$, $(5, 8, 7)$ are collinear.

Answer 4:

The given points are A$(2, 3, 4)$, B$(-1, -2, 1)$, and C$(5, 8, 7)$.

It is known that the direction ratios of line joining the points, (x_1, y_1, z_1) and (x_2, y_2, z_2), are given by,

$$x_2 - x_1, y_2 - y_1, \text{and } z_2 - z_1 .$$

The direction ratios of AB are $(-1 - 2)$, $(-2 - 3)$, and $(1 - 4)$ i.e., -3, -5, and -3.

The direction ratios of BC are $(5 - (-1))$, $(8 - (-2))$, and $(7 - 1)$ i.e., 6, 10, and 6.

It can be seen that the direction ratios of BC are -2 times that of AB i.e., they are proportional.

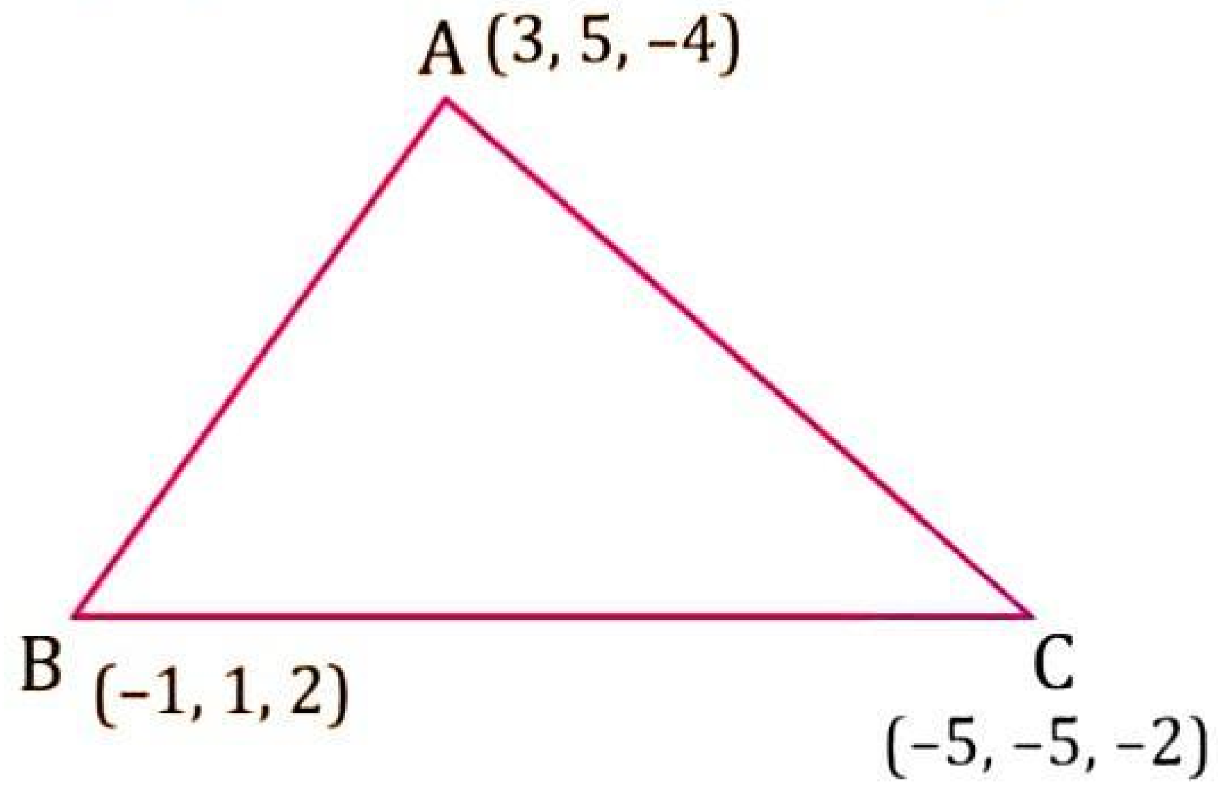
Therefore, AB is parallel to BC. Since point B is common to both AB and BC, points A, B, and C are collinear.

Question 5:

Find the direction cosines of the sides of the triangle whose vertices are $(3, 5, -4)$, $(-1, 1, 2)$ and $(-5, -5, -2)$.

Answer 5:

The vertices of ΔABC are A $(3, 5, -4)$, B $(-1, 1, 2)$, and C $(-5, -5, -2)$.



The direction ratios of side AB are $(-1 - 3)$, $(1 - 5)$, and $(2 - (-4))$ i.e., -4, -4, and 6.

Therefore, the direction cosines of AB are

$$= \frac{-4}{\sqrt{(-4)^2 + (-4)^2 + (6)^2}}, \frac{-4}{\sqrt{(-4)^2 + (-4)^2 + (6)^2}}, \frac{6}{\sqrt{(-4)^2 + (-4)^2 + (6)^2}}$$

$$= \frac{-4}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}, \frac{6}{2\sqrt{17}} \quad = \frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$$

The direction ratios of BC are $(-5 - (-1))$, $(-5 - 1)$, and $(-2 - 2)$ i.e., -4, -6, and -4.

Therefore, the direction cosines of BC are

$$= \frac{-4}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}}, \frac{-4}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}}, \frac{6}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}}$$

$$= \frac{-4}{2\sqrt{17}}, \frac{-6}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}$$

The direction ratios of CA are $(-5 - 3)$, $(-5 - 5)$, and $(-2 - (-4))$ i.e., -8, -10, and 2.

Therefore, the direction cosines of AC are

$$= \frac{-8}{\sqrt{(-8)^2 + (10)^2 + (2)^2}}, \frac{-5}{\sqrt{(-8)^2 + (10)^2 + (2)^2}}, \frac{2}{\sqrt{(-8)^2 + (10)^2 + (2\ )^2}}$$

$$= \frac{-8}{2\sqrt{42}}, \frac{-10}{2\sqrt{42}}, \frac{2}{2\sqrt{42}} = \frac{-4}{\sqrt{42}}, \frac{-5}{\sqrt{42}}, \frac{1}{\sqrt{42}}$$

Mathematics

(Chapter – 11) (Three Dimensional Geometry) (Exercise 11.2)
(Class – XII)

Question 1:

Show that the three lines with direction cosines $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

Answer 1:

Two lines with direction cosines, l_1, m_1, n_1 *and* l_2, m_2, n_2 are perpendicular to each other, if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

(i) For the lines with direction cosines

$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ and $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = \frac{12}{13} \times \frac{4}{13} + \left(\frac{-3}{13}\right) \times \frac{12}{13} + \left(\frac{-4}{13}\right) \times \frac{3}{13} = \frac{48}{169} - \frac{36}{169} - \frac{12}{169} = 0$$

Therefore, the lines are perpendicular.

(ii) For the lines with direction cosines,

$\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = \frac{4}{13} \times \frac{3}{13} + \frac{12}{13} \times \left(\frac{-4}{13}\right) + \frac{3}{13} \times \frac{12}{13} = 0$$

Therefore, the lines are perpendicular.

(iii) For the lines with direction cosines,

$\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ and $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$, we get

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = \left(\frac{3}{13}\right) \times \left(\frac{12}{13}\right) + \left(\frac{-4}{13}\right) \times \left(\frac{-3}{13}\right) + \left(\frac{12}{13}\right) \times \left(\frac{-4}{13}\right) = 0$$

Therefore, the lines are perpendicular.

Thus, all the lines are mutually perpendicular.

Question 2:

Show that the line through the points (1, −1, 2), (3, 4, −2) is perpendicular to the line through the points (0, 3, 2), (3, 5, 6).

Answer 2:

Let AB be the line joining the points, (1, −1, 2) and (3, 4, − 2), and CD be the line joining the points, (0, 3, 2) and (3, 5, 6).

The direction ratios a_1, b_1, c_1 of AB are (3 – 1), (4 – (−1)), and (−2 – 2) i.e., 2, 5, and −4.

The direction ratios, a_2, b_2, c_2 of CD are (3 – 0), (5 – 3), and (6 −2) i.e., 3, 2, and 4.

AB and CD will be perpendicular to each other, if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

$a_1 a_2 + b_1 b_2 + c_1 c_2$

$= 2 \times 3 + 5 \times 2 + (-4) \times 4$

$= 6 + 10 - 16$

$= 0$

Therefore, AB and CD are perpendicular to each other.

Question 3:

Show that the line through the points (4, 7, 8), (2, 3, 4) is parallel to the line through the points (−1, −2, 1), (1, 2, 5).

Answer 3:

Let AB be the line through the points, (4, 7, 8) and (2, 3, 4), and CD be the line through the points, (−1, −2, 1) and (1, 2, 5). The directions ratios, a_1, b_1, c_1, of AB are (2 − 4), (3 − 7), and (4 − 8) i.e., −2, −4, and−4.

The direction ratios, a_2, b_2, c_2 of CD are (1 − (−1)), (2 − (−2)), and (5 − 1) i.e., 2, 4,and 4.

AB will be parallel to CD, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Here, $\frac{a_1}{a_2} = \frac{-2}{2} = -1,\quad \frac{b_1}{b_2} = \frac{-4}{4} = -1$ and $\frac{c_1}{c_2} = \frac{-4}{4} = -1$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Thus, AB is parallel to CD.

Question 4:

Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$.

Answer 4:

It is given that the line passes through the point A (1, 2, 3). Therefore, the position vector through A is

$\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

The given vector: $\vec{b} = 3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

It is known that the line which passes through point $\vec{a}$ and parallel to $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a constant. Therefore,

$$\vec{r} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k} + \lambda(3\hat{\imath} + 2\hat{\jmath} - 2\hat{k})$$

This is the required equation of the line.

Question 5:

Find the equation of the line in vector and in Cartesian form that passes through the point with position vector $2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ and is in the direction $\hat{\imath} + 2\hat{\jmath} - \hat{k}$

Answer 5:

It is given that the line passes through the point with position vector

$\vec{a} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ ... (1)

The given vector:

$\vec{b} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$ ... (2)

It is known that a line through a point with position vector $\vec{a}$ and parallel to $\vec{b}$ is given by

$\vec{r} = \vec{a} + \lambda\vec{b} \quad \Rightarrow \vec{r} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k} + \lambda(\hat{\imath} + 2\hat{\jmath} - \hat{k})$

This is the required equation of the line in vector form.

Now, let $\vec{r} = x\hat{\imath} - y\hat{\jmath} + z\hat{k} \Rightarrow x\hat{\imath} - y\hat{\jmath} + z\hat{k} = (\lambda + 2)\hat{\imath} + (2\lambda - 1)\hat{\jmath} + (-\lambda + 4)\hat{k}$

Comparing the coefficient to eliminate λ, we get the Cartesian form equation as

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

This is the required equation of the given line in Cartesian form.

Question 6:

Find the Cartesian equation of the line which passes through the point (−2, 4, −5) and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.

Answer 6:

It is given that the line passes through the point $(-2, 4, -5)$ and is parallel to $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$

The direction ratios of the line $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$, are 3, 5 and 6.

The required line is parallel to $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$

Therefore, its direction ratios are 3k, 5k, and 6k, where $k \neq 0$

It is known that the equation of the line through the point (x_1, y_1, z_1) and with direction ratios, a, b, c, is given by:

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

Therefore, the equation of the required line is $\frac{x+2}{3k}=\frac{y-4}{5k}=\frac{z+5}{6k} \Rightarrow \frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}=k$

Question 7:

The Cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Write its vector form.

Answer 7:

The Cartesian equation of the line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$.

The given line passes through the point $(5, -4, 6)$. The position vector of this point is $\vec{a}=5\hat{\imath}-4\hat{\jmath}+6\hat{k}$.

Also, the direction ratios of the given line are 3, 7, and 2.

This means that the line is in the direction of vector, $\vec{b}=3\hat{\imath}+7\hat{\jmath}+2\hat{k}$

It is known that the line through position vector $\vec{a}$ and in the direction of the vector $\vec{b}$ is given by the equation:

$\vec{r}=\vec{a}+\lambda\vec{b}$, where $\lambda \in R \Rightarrow \vec{r}=(5\hat{\imath}-4\hat{\jmath}+6\hat{k})+\lambda(3\hat{\imath}+7\hat{\jmath}+2\hat{k})$

This is the required equation of the given line in vector form.

Question 8:

Find the angle between the following pairs of lines:

(i) $\vec{r}=2\hat{\imath}-5\hat{\jmath}+\hat{k}+\lambda(3\hat{\imath}+2\hat{\jmath}+6\hat{k})$ and $\vec{r}=7\hat{\imath}-6\hat{k}+\mu(\hat{\imath}+2\hat{\jmath}+2\hat{k})$

(ii) $\vec{r}=3\hat{\imath}+\hat{\jmath}-2\hat{k}+\lambda(\hat{\imath}-\hat{\jmath}-2\hat{k})$ and $\vec{r}=2\hat{\imath}-\hat{\jmath}-56\hat{k}+\mu(3\hat{\imath}-5\hat{\jmath}-4\hat{k})$

Answer 8:

(i) Let Q be the angle between the given lines.

The angle between the given pairs of lines is given by $\cos Q=\left|\frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|$

The given lines are parallel to the vectors $\vec{b_1}=3\hat{\imath}+2\hat{\jmath}+6\hat{k}$ and $\vec{b_2}=\hat{\imath}+2\hat{\jmath}+2\hat{k}$ respectively.

Now,

$|\vec{b_1}|=\sqrt{(3)^2+(2)^2+(6)^2}=7$ and $|\vec{b_2}|=\sqrt{(1)^2+(2)^2+(2)^2}=3$

$\vec{b_1}.\vec{b_2}=(3\hat{\imath}+2\hat{\jmath}+6\hat{k}).(\hat{\imath}+2\hat{\jmath}+2\hat{k}) \Rightarrow \vec{b_1}.\vec{b_2}=3\times1+2\times2+6\times2=3+4+12=19$

Therefore,

$\cos Q=\frac{19}{7\times3} \Rightarrow Q=cos^{-1}\left(\frac{19}{21}\right)$

(ii) The given lines are parallel to the vectors, $\vec{b_1}=\hat{\imath}-\hat{\jmath}-2\hat{k}$ and $\vec{b_2}=3\hat{\imath}-5\hat{\jmath}-4\hat{k}$ respectively.

$|\vec{b_1}|=\sqrt{(1)^2+(-1)^2+(-2)^2}=\sqrt{6}$ and $|\vec{b_2}|=\sqrt{(3)^2+(-5)^2+(-4)^2}=\sqrt{50}=5\sqrt{2}$

$\vec{b_1}.\vec{b_2}=(\hat{\imath}-\hat{\jmath}-2\hat{k}).(3\hat{\imath}-5\hat{\jmath}-4\hat{k})=1.3-1(-5)-2(-4)=3+5+8=16$

Therefore,

$$\cos Q=\left|\frac{\vec{b_1}.\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|=\frac{16}{\sqrt{6}.5\sqrt{2}}=\frac{16}{\sqrt{3}.\sqrt{2}.5\sqrt{2}}=\frac{16}{10\sqrt{3}}=\frac{8}{5\sqrt{3}} \Rightarrow Q=cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

Question 9:

Find the angle between the following pairs of lines:

(i) $\frac{x-2}{2}=\frac{y-1}{5}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$ **(ii)** $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x-5}{4}=\frac{y-2}{1}=\frac{z-3}{8}$

Answer 9:

(i) Let $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$ be the vectors parallel to the pair of lines,

$\frac{x-2}{2}=\frac{y-1}{5}=\frac{z+3}{-3}$ and $\frac{x+2}{-3}=\frac{y-4}{8}=\frac{z+5}{4}$

$\overrightarrow{b_1}=2\hat{\imath}+5\hat{\jmath}-3\hat{k}$ and $\overrightarrow{b_2}=-\hat{\imath}+8\hat{\jmath}+4\hat{k}$

$|\overrightarrow{b_1}|=\sqrt{(2)^2+(5)^2+(-3)^2}=\sqrt{38}$

$|\overrightarrow{b_2}|=\sqrt{(-1)^2+(8)^2+(4)^2}=\sqrt{81}=9$

$\overrightarrow{b_1}.\overrightarrow{b_2}=(2\hat{\imath}+5\hat{\jmath}-3\hat{k}).(-\hat{\imath}+8\hat{\jmath}+4\hat{k})=2(-1)+5\times 8+(-3).4=-2+40-12=26$

The angle, Q, between the given pair of lines is given by the relation,

$$\cos Q=\left|\frac{\overrightarrow{b_1}.\overrightarrow{b_2}}{|\overrightarrow{b_1}||\overrightarrow{b_2}|}\right|$$

Therefore, $\cos Q=\frac{26}{9\sqrt{38}} \Rightarrow Q=cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$

(ii) Let $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$ be the vectors parallel to the given pair of lines,

$\frac{x-5}{4}=\frac{y-5}{1}=\frac{z-3}{8}$ and $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$

Therefore, $\overrightarrow{b_1}=2\hat{\imath}+2\hat{\jmath}+\hat{k}$ and $\overrightarrow{b_2}=4\hat{\imath}+\hat{\jmath}+8\hat{k}$

Now,

$|\overrightarrow{b_1}|=\sqrt{(2)^2+(2)^2+(1)^2}=\sqrt{9}=3$

$|\overrightarrow{b_2}|=\sqrt{(4)^2+(1)^2+(8)^2}=\sqrt{81}=9$

$\overrightarrow{b_1}.\overrightarrow{b_2}=(2\hat{\imath}+2\hat{\jmath}+\hat{k}).(4\hat{\imath}+\hat{\jmath}+8\hat{k})=2\times 4+2\times 1+1\times 8=8+2+8=18$

If Q is the angle between the given pair of lines, then

$\cos Q=\left|\frac{\overrightarrow{b_1}.\overrightarrow{b_2}}{|\overrightarrow{b_1}||\overrightarrow{b_2}|}\right|=\frac{18}{3\times 9}=\frac{2}{3} \qquad \Rightarrow Q=cos^{-1}\left(\frac{2}{3}\right)$

Question 10:

Find the values of p so the lines $\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{2}$ and $\frac{7-7x}{3p}=\frac{y-5}{2}=\frac{6-z}{5}$ are at right angles.

Answer 10:

The given equations can be written in the standard form as

$\frac{x-1}{-3}=\frac{y-2}{\frac{2p}{7}}=\frac{z-3}{2}$ and $\frac{x-1}{\frac{-3p}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}$

The direction ratios of the lines are $-3,\frac{2p}{7},2$ and $\frac{-3p}{7},1,-5$ respectively.

Two lines with direction ratios, a_1,b_1,c_1 and a_2,b_2,c_2 are perpendicular to each other, if

$$a_1.a_2+b_1.b_2+c_1.c_2=0$$

$\Rightarrow(-3).\left(\frac{-3p}{7}\right)+\left(\frac{2p}{7}\right).(1)+(2).(-5)=0 \qquad \Rightarrow\frac{9p}{7}+\frac{2p}{7}=10 \qquad \Rightarrow 11p=70 \qquad \Rightarrow p=\frac{70}{11}$

Thus, the required value of p is $\frac{70}{11}$.

Question 11:

Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other.

Answer 11:

The direction vector of $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ is $(7, -5, 1)$ and direction vector of

$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

is $(1, 2, 3)$

Dot product of the vectors is $7 \times 1 + (-5)(2) + 1 \times 3 = 0$

Therefore the lines are perpendicular.

Question 12:

Find the shortest distance between the lines
$\vec{r} = (\hat{\imath} + 2\hat{\jmath} + \hat{k}) + \lambda(\hat{\imath} - \hat{\jmath} + \hat{k})$ and $\vec{r} = 2\hat{\imath} - \hat{\jmath} - \hat{k} + \mu(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$.

Answer 12:

The equations of the given lines are

$\vec{r} = (\hat{\imath} + 2\hat{\jmath} + \hat{k}) + \lambda(3\hat{\imath} - 2\hat{\jmath} + 6\hat{k})$ and $\vec{r} = 2\hat{\imath} - \hat{\jmath} - \hat{k} + \mu(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$

It is known that the shortest distance between the lines, $\vec{r} = \overrightarrow{a_1} + \lambda \overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu \overrightarrow{b_2}$ is given by

$$d = \left| \frac{(\overrightarrow{b_1} \times \overrightarrow{b_2}).(\overrightarrow{a_2} - \overrightarrow{a_1})}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|} \right| \quad \dots (1)$$

Comparing the given equations, we get

$\overrightarrow{a_1} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$

$\overrightarrow{b_1} = \hat{\imath} - \hat{\jmath} + \hat{k}$

$\overrightarrow{a_2} = 2\hat{\imath} - \hat{\jmath} - \hat{k}$

$\overrightarrow{b_2} = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$

Now, $\overrightarrow{a_2} - \overrightarrow{a_1} = (2\hat{\imath} - \hat{\jmath} + \hat{k}) - (\hat{\imath} + 2\hat{\jmath} + \hat{k}) = \hat{\imath} - 3\hat{\jmath} - 2\hat{k}$, therefore

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = (-2-1)\hat{\imath} - (2-2)\hat{\jmath} + (1+2)\hat{k} = -3\hat{\imath} + 3\hat{k}$$

$\Rightarrow |\overrightarrow{b_1} \times \overrightarrow{b_2}| = \sqrt{(-3)^2 + (3)^2}$

$= \sqrt{9+9} = 3\sqrt{2}$

Substituting all the values in equation (1), we get

$$d = \left| \frac{(-3\hat{\imath} + 3\hat{k}).(\hat{\imath} - 3\hat{\jmath} - 2\hat{k})}{3\sqrt{2}} \right| = \left| \frac{-3.1 + 3(-2)}{3\sqrt{2}} \right| = \left| \frac{-9}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the two lines is $\frac{3\sqrt{2}}{2}$ units.

Question 13:

Find the shortest distance between the lines $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$.

Answer 13:

Equeations of the given lines $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

It is known that the shortest distance between the two lines,

$\frac{x+x_1}{a_1}=\frac{y+y_1}{b_1}=\frac{z+z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is given by

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}} \quad \ldots(1)$$

Comparing the given equations, we have

$x_1=-1,\quad y_1=-1\quad z_1=-1$

$a_1=7\quad b_1=-6\quad c_1=1$

$x_2=3\quad y_2=5\quad z_2=7$

$a_2=1\quad b_2=-2\quad c_2=1$

Then

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=\begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$=4(-6+2)-6(7-1)+8(-14+6)=-16-36-64=-116$

and $\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}=\sqrt{(-6+2)^2+(1+7)^2+(-14+6)^2}$

$=\sqrt{16+36+64}\ =\sqrt{116}\ =2\sqrt{29}$

Substituting all the values in equation (1), we have

$$d=\frac{-116}{2\sqrt{29}}=\frac{-58}{\sqrt{29}}=-2\sqrt{29}$$

Since, distance is always non-negative, so, the distance between the given lines is $2\sqrt{29}$ units.

Question 14:

Find the shortest distance between the lines whose vector equations are

$\vec{r}=(\hat{\imath}+2\hat{\jmath}+3\hat{k})+\lambda(\hat{\imath}-3\hat{\jmath}+2\hat{k})$ and $\vec{r}=4\hat{\imath}+5\hat{\jmath}+6\hat{k}+\mu(2\hat{\imath}+3\hat{\jmath}+\hat{k})$.

Answer 14:

$\vec{r}=(\hat{\imath}+2\hat{\jmath}+3\hat{k})+\lambda(\hat{\imath}-3\hat{\jmath}+2\hat{k})$ and $\vec{r}=4\hat{\imath}+5\hat{\jmath}+6\hat{k}+\mu(2\hat{\imath}+4\hat{\jmath}+\hat{k})$

It is known that the shortest distance between the lines $\vec{r}=\overrightarrow{a_1}+\lambda\,\overrightarrow{b_1}$ *and* $\vec{r}=\overrightarrow{a_2}+\mu\,\overrightarrow{b_2}$ is given by

$$d=\left|\frac{(\overrightarrow{b_1}\times\overrightarrow{b_2}).(\overrightarrow{a_2}-\overrightarrow{a_1})}{|\overrightarrow{b_1}\times\overrightarrow{b_2}|}\right| \quad \ldots(1)$$

Comparing the given equations with $\vec{r}=\overrightarrow{a_1}+\lambda\,\overrightarrow{b_1}$ and $\vec{r}=\overrightarrow{a_2}+\mu\,\overrightarrow{b_2}$, we have

$\overrightarrow{a_1}=\hat{\imath}+2\hat{\jmath}+3\hat{k}$

$\overrightarrow{b_1}=\hat{\imath}-3\hat{\jmath}+2\hat{k}$

$\overrightarrow{a_2}=4\hat{\imath}+5\hat{\jmath}+6\hat{k}$

$\overrightarrow{b_2}=2\hat{\imath}+3\hat{\jmath}+\hat{k}$

Therefore, $\overrightarrow{a_2}-\overrightarrow{a_1}=(4\hat{\imath}+5\hat{\jmath}+6\hat{k})-(\hat{\imath}+2\hat{\jmath}+3\hat{k})=3\hat{\imath}+3\hat{\jmath}+3\hat{k}$ and

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-3-6)\hat{\imath} - (1-4)\hat{\jmath} + (3+6)\hat{k} = -9\hat{\imath} + 3\hat{\jmath} + 9\hat{k}$$

$$\Rightarrow \left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81+9+81} = 3\sqrt{19}$$

$$\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right).\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right) = \left(-9\hat{\imath} + 3\hat{\jmath} + 9\hat{k}\right).\left(3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}\right) = -9 \times 3 + 3 \times 3 + 9 \times 3 = 9$$

Substituting all the values in equation (1), we get

$$d = \left|\frac{9}{3\sqrt{19}}\right| = \frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the two given lines is $\frac{3}{\sqrt{19}}$ units.

Question 15:

Find the shortest distance between the lines whose vector equations are

$\vec{r} = (1-t)\hat{\imath} + (t-2)\hat{\jmath} + (3-2t)\hat{k}$ and $\vec{r} = (s+1)\hat{\imath} + (2s-1)\hat{\jmath} - (2s+1)\hat{k}$.

Answer 15:

The given lines are

$$\vec{r} = (1-t)\hat{\imath} + (t-2)\hat{\jmath} + (3-2t)\hat{k}$$

$$\Rightarrow \vec{r} = \left(\hat{\imath} - 2\hat{\jmath} + 3\hat{k}\right) + t\left(-\hat{\imath} + \hat{\jmath} - 2\hat{k}\right) \quad \ldots (1)$$

$$\vec{r} = (s+1)\hat{\imath} + (2s-1)\hat{\jmath} - (2s+1)\hat{k}$$

$$\Rightarrow \vec{r} = \left(\hat{\imath} - \hat{\jmath} + \hat{k}\right) + s\left(\hat{\imath} + 2\hat{\jmath} - 2\hat{k}\right) \quad \ldots (2)$$

It is known that the shortest distance between the lines $\vec{r} = \overrightarrow{a_1} + \lambda \overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu \overrightarrow{b_2}$ is given by

$$d = \left|\frac{\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right).\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right)}{\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right|}\right| \quad \ldots (3)$$

For the given equations,

$\overrightarrow{a_1} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$, $\overrightarrow{b_1} = -\hat{\imath} + \hat{\jmath} - 2\hat{k}$, $\overrightarrow{a_2} = \hat{\imath} - \hat{\jmath} - \hat{k}$ and $\overrightarrow{b_2} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

Therefore,

$$\overrightarrow{a_2} - \overrightarrow{a_1} = \left(\hat{\imath} - \hat{\jmath} - \hat{k}\right) - \left(\hat{\imath} - 2\hat{\jmath} + 3\hat{k}\right) = \hat{\jmath} - 4\hat{k}$$

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = (-2+4)\hat{\imath} - (2+2)\hat{\jmath} + (-2-1)\hat{k} = 2\hat{\imath} - 4\hat{\jmath} - 3\hat{k}$$

$$\Rightarrow \left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{(2)^2 + (-4)^2 + (-3)^2} = \sqrt{4+16+9} = \sqrt{29}$$

$$\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right).\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right) = \left(2\hat{\imath} - 4\hat{\jmath} - 3\hat{k}\right).\left(\hat{\jmath} - 4\hat{k}\right) = -4 + 12 = 8$$

Substituting all the values in equation (3), we get

$$d = \left|\frac{8}{\sqrt{29}}\right|$$

Therefore, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.

Mathematics

(Chapter – 11) (Three Dimensional Geometry) (Miscellaneous Exercise)
(Class – XII)

Question 1:
Find the angle between the lines whose direction ratios are a, b, c and b − c, c − a, a −b.

Answer 1:
The angle Q between the lines with direction cosines, a, b, c and b − c, c − a, a − b, is given by,

$$\cos Q = \left|\frac{a(b-c)+b(c-a)+c(a-b)}{\sqrt{a^2+b^2+c^2}+\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}\right|$$

$$= \left|\frac{ab-ac+bc-ab+ac-bc}{\sqrt{a^2+b^2+c^2}+\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}\right|$$

$$= \left|\frac{0}{\sqrt{a^2+b^2+c^2}+\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}\right|$$

$\Rightarrow \cos Q = 0 \quad \Rightarrow Q = cos^{-1}0 \quad \Rightarrow Q = 90^0$

Thus, the angle between the lines is 90°.

Question 2:
Find the equation of a line parallel to x-axis and passing through the origin.

Answer 2:
The line parallel to x-axis and passing through the origin is x-axis itself.
Let A be a point on x-axis. Therefore, the coordinates of A are given by $(a, 0, 0)$, where $a \in R$
Direction ratios of OA are $(a - 0)$, 0, 0 = a, 0, 0
The equation of OA is given by,

$$\frac{x-0}{a} = \frac{y-0}{0} = \frac{z-0}{0} \qquad \Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{0}$$

Thus, the equation of line parallel to x − axis and passing through origin is $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$.

Question 3:
If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k.

Answer 3:
The direction of ratios of the lines, $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are -3, 2k, 2 and 3k, 1, -5 respectively.

It is known that two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular, if

$a_1a_2 + b_1b_2 + c_1c_2 = 0 \quad \Rightarrow -3(3k) + 2k \times 1 + 2(-5) = 0 \quad \Rightarrow -9k + 2k - 10 = 0$

$$\Rightarrow 7k = -10 \Rightarrow k = -\frac{10}{7}$$

Therefore, for $k = -\frac{10}{7}$, the given lines are perpendicular to each other.

Question 4:
Find the shortest distance between lines $\vec{r} = 6\hat{\imath} + 2\hat{\jmath} + 2\hat{k} + \lambda\left(\hat{\imath} - 2\hat{\jmath} + 2\hat{k}\right)$ and $\vec{r} = -4\hat{\imath} - \hat{k} + \mu\left(3\hat{\imath} - 2\hat{\jmath} - 2\hat{k}\right)$.

Answer 4:
The given lines are

$\vec{r} = 6\hat{\imath} + 2\hat{\jmath} + 2\hat{k} + \lambda\left(\hat{\imath} - 2\hat{\jmath} + 2\hat{k}\right) \quad \ldots (1)$

$\vec{r} = -4\hat{\imath} - \hat{k} + \mu\left(3\hat{\imath} - 2\hat{\jmath} - 2\hat{k}\right) \quad \ldots (2)$

It is known that the shortest distance between two lines, $\vec{r} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \lambda\overrightarrow{b_2}$ is given by

$$d = \left|\frac{(\overrightarrow{b_1} \times \overrightarrow{b_2}).(\overrightarrow{a_2} - \overrightarrow{a_1})}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|}\right| \qquad \ldots (3)$$

Comparing $\vec{r} = \overrightarrow{a_1} + \lambda \overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \lambda \overrightarrow{b_2}$ to equations (1) and (2), we have

$\overrightarrow{a_1} = 6\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$
$\overrightarrow{b_1} = \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$
$\overrightarrow{a_2} = -4\hat{\imath} - \hat{k}$
$\overrightarrow{b_2} = 3\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$

$\Rightarrow \overrightarrow{a_2} - \overrightarrow{a_1} = \left(-4\hat{\imath} - \hat{k}\right) - \left(6\hat{\imath} + 2\hat{\jmath} + 2\hat{k}\right) = -10\hat{\imath} - 2\hat{\jmath} - 3\hat{k}$

$\Rightarrow |\overrightarrow{b_1} \times \overrightarrow{b_2}| = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = (4+4)\hat{\imath} - (-2-6)\hat{\jmath} + (-2+6)\hat{k} = 8\hat{\imath} + 8\hat{\jmath} + 4\hat{k}$

$\Rightarrow |\overrightarrow{b_1} \times \overrightarrow{b_2}| = \sqrt{(8)^2 + (8)^2 + (4)^2} = 12$

$\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right).\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right) = \left(8\hat{\imath} + 8\hat{\jmath} + 4\hat{k}\right).\left(-10\hat{\imath} - 2\hat{\jmath} - 3\hat{k}\right) = -80 - 16 - 12 = -108$

Substituting all the values in equation (3), we have

$$d = \left|\frac{-108}{12}\right| = 9$$

Therefore, the shortest distance between the two given lines is 9 units.

Question 5:

Find the vector equation of the line passing through the point (1, 2, − 4) and perpendicular to the two lines:

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \quad \text{and} \quad \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

Answer 5:

Let the required line be parallel to the vector $\vec{b}$ given by $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$.

The position vector of the point (1, 2, − 4) is $\vec{a} = \hat{\imath} + 2\hat{\jmath} - 4\hat{k}$.

The equation of the line passing through (1, 2, −4) and parallel to vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$

$\Rightarrow \vec{r}\ \left(\hat{\imath} + 2\hat{\jmath} - 4\hat{k}\right) + \lambda\left(b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}\right) \quad \ldots (1)$

The equations of the lines are

$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \quad \ldots (2)$

$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad \ldots (3)$

Line (1) and line (2) are perpendicular to each other. Therefore,

$3b_1 - 16b_2 + 7b_3 = 0 \quad \ldots (4)$

Also, line (1) and line (3) are perpendicular to each other. Therefore,

$3b_1 + 8b_2 - 5b_3 = 0 \quad \ldots (5)$

From equations (4) and (5), we have

$$\frac{b_1}{(-16)(5) - 8 \times 7} = \frac{b_2}{7 \times 3 - 3(-5)} = \frac{b_3}{3 \times 8 - 3(-16)} \quad \Rightarrow \frac{b_1}{24} = \frac{b_2}{36} = \frac{b_3}{72} \quad \Rightarrow \frac{b_1}{2} = \frac{b_2}{3} = \frac{b_3}{6}$$

$\Rightarrow$ Direction ratios of $\vec{b}$ are 2, 3 and 6 or $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$.

Substituting $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$ in equation (1), we have $\vec{r}\ \left(\hat{\imath} + 2\hat{\jmath} - 4\hat{k}\right) + \lambda\left(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}\ \right)$.

Hence, this is the equation of the required line.