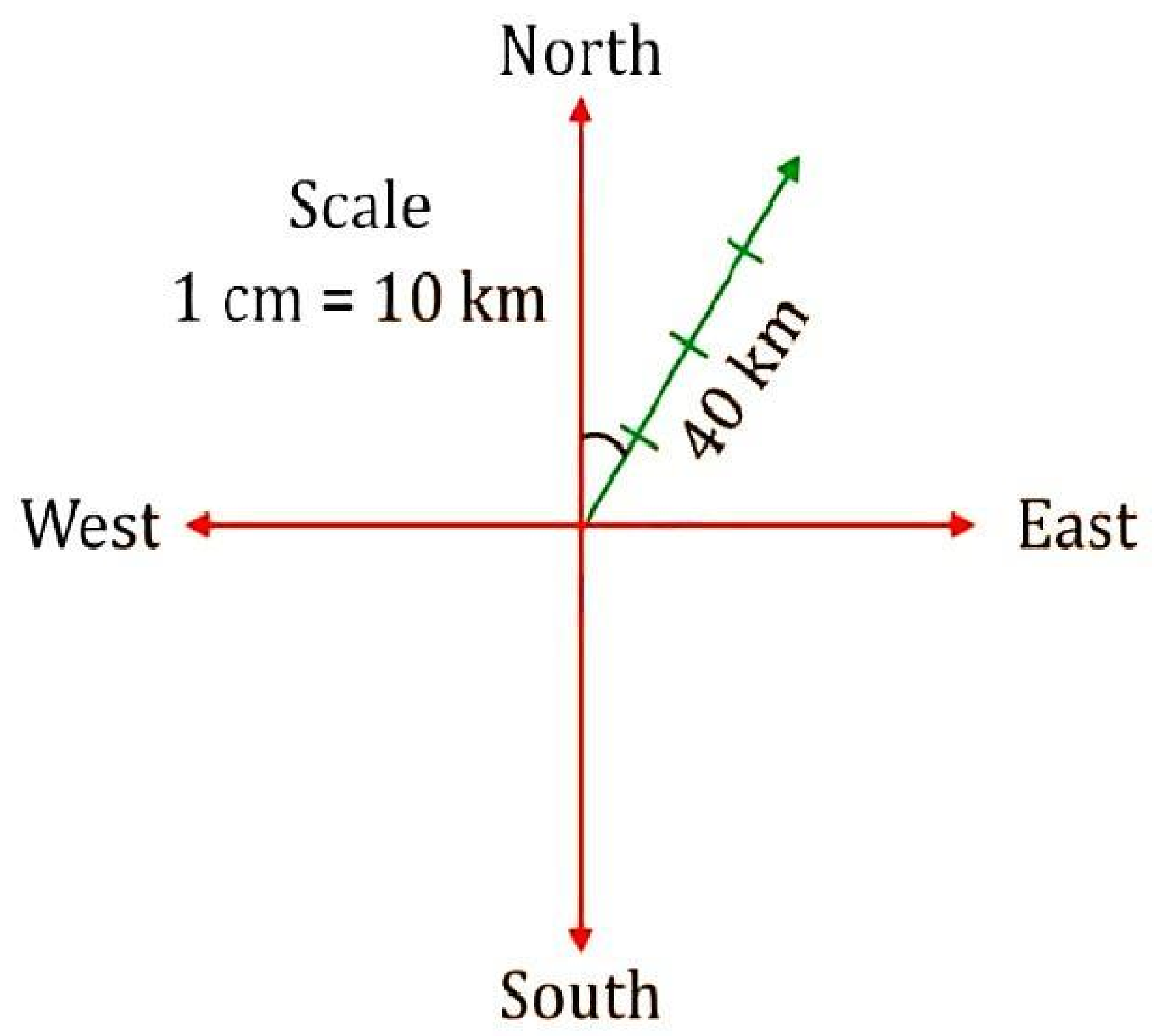
Mathematics

(Chapter – 10) (Vector Algebra) (Exercise 10.1)
(Class – XII)

Question 1:
Represent graphically a displacement of 40 km, 30° east of north.

Answer 1:
Taking scale 10 km = 1 unit
Here, the vector $\overrightarrow{OP}$ represent the displacement of 40 km, 30° East of North.



Question 2:
Classify the following measures as scalars and vectors.

(i) 10 kg
(ii) 2 metres north-west
(iii) 40°
(iv) 40 watt
(v) 10^{-19} coulomb
(vi) 20 m/s^2

Answer 2:
(i) 10 kg is a scalar quantity because it involves only magnitude.
(ii) 2 meters north-west is a vector quantity as it involves both magnitude and direction.
(iii) 40° is a scalar quantity as it involves only magnitude.
(iv) 40 watts is a scalar quantity as it involves only magnitude.
(v) 10^{-19} coulomb is a scalar quantity as it involves only magnitude.
(vi) 20 m/s^2 is a vector quantity as it involves magnitude as well as direction.

Question 3:
Classify the following as scalar and vector quantities.

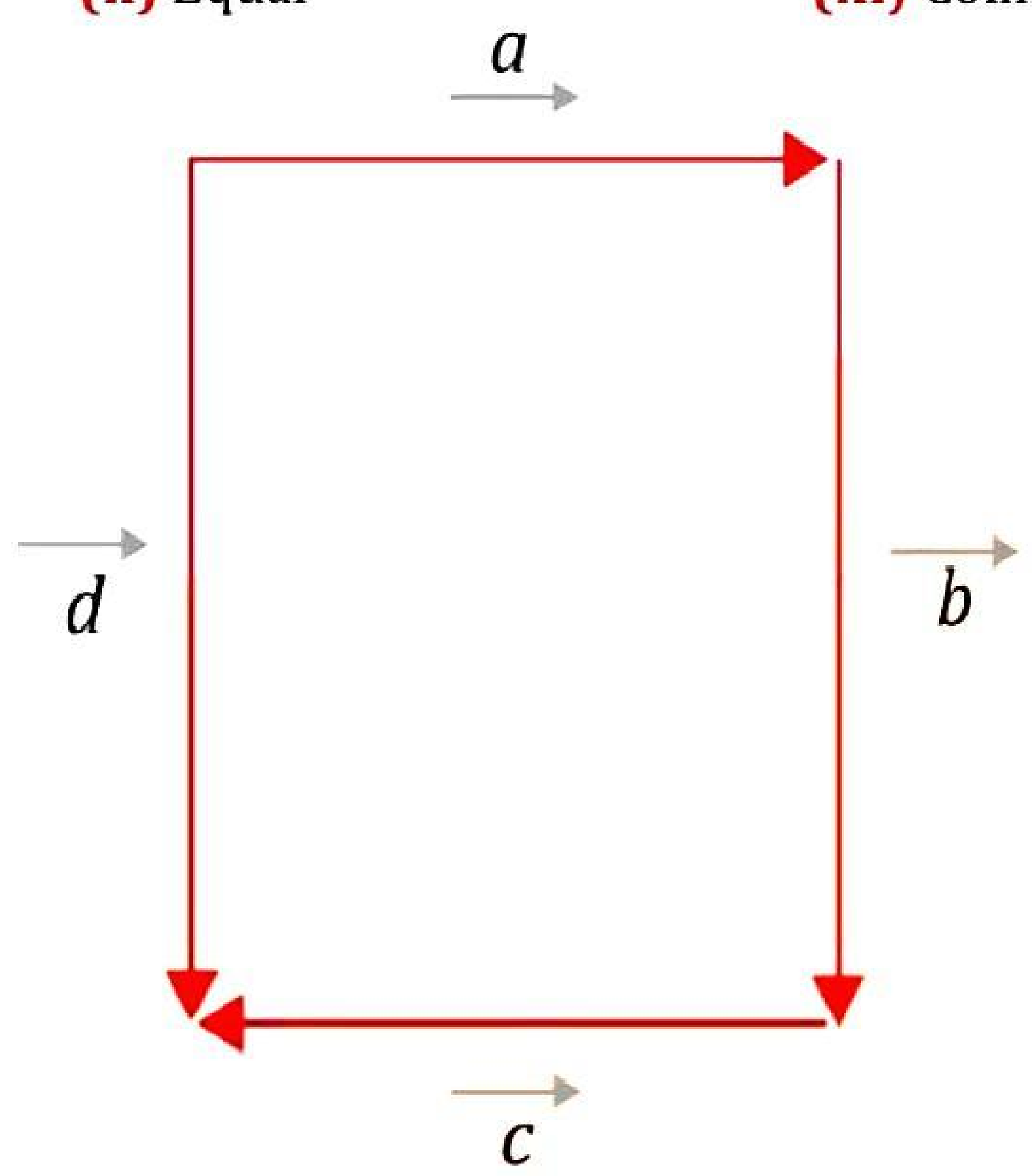
(i) time period
(ii) distance
(iii) force
(iv) velocity
(v) work done

Answer 3:
(i) Time period is a scalar quantity as it involves only magnitude.
(ii) Distance is a scalar quantity as it involves only magnitude.
(iii) Force is a vector quantity as it involves both magnitude and direction.
(iv) Velocity is a vector quantity as it involves both magnitude as well as direction.
(v) Work done is a scalar quantity as it involves only magnitude.

Question 4:

In Figure, identify the following vectors.

(i) Coinitial **(ii)** Equal **(iii)** Collinear but not equal



Answer 4:

(i) Vectors $\vec{a}$ and $\vec{d}$ are coinitial because they have the same initial point.

(ii) Vectors $\vec{d}$ and $\vec{b}$ are equal because they have the same magnitude and direction.

(iii) Vectors $\vec{a}$ and $\vec{c}$ are collinear but not equal. This is because although they are parallel, but their directions are not the same.

Question 5:

Answer the following as true or false.

(i) $\vec{a}$ and $-\vec{a}$ are collinear.

(ii) Two collinear vectors are always equal in magnitude.

(iii) Two vectors having same magnitude are collinear.

(iv) Two collinear vectors having the same magnitude are equal.

Answer 5:

(i) True.

Vectors $\vec{a}$ and $-\vec{a}$ are parallel to the same line.

(ii) False.

Collinear vectors are those vectors that are parallel to the same line. But it is not necessary that they are equal also.

(iii) False.

Two vectors having same magnitude may have different direction, so they are not collinear.

(iv) False

Two collinear vectors having same magnitude are not equal when they are opposite in directions.

Mathematics

(Chapter – 10) (Vector Algebra) (Exercise 10.2)
(Class – XII)

Question 1:

Compute the magnitude of the following vectors:

$$\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}; \qquad \vec{b} = 2\hat{\imath} - 7\hat{\jmath} - 3\hat{k}; \qquad \vec{c} = \frac{1}{\sqrt{3}}\hat{\imath} + \frac{1}{\sqrt{3}}\hat{\jmath} - \frac{1}{\sqrt{3}}\hat{k}$$

Answer 1:

The given vectors are:

$$\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}; \qquad \vec{b} = 2\hat{\imath} - 7\hat{\jmath} - 3\hat{k}; \qquad \vec{c} = \frac{1}{\sqrt{3}}\hat{\imath} + \frac{1}{\sqrt{3}}\hat{\jmath} - \frac{1}{\sqrt{3}}\hat{k},$$

Therefore, we have

$$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$|\vec{b}| = \sqrt{2^2 + (-7)^2 + (-3)^2} = \sqrt{4 + 49 + 9} = \sqrt{62}$$

$$|\vec{c}| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \sqrt{\frac{3}{3}} = 1$$

Question 2:

Write two different vectors having same magnitude.

Answer 2:

Consider $\vec{a}$ and $\vec{b}$ two different vectors as follows:

$\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ and $\vec{b} = 2\hat{\imath} + \hat{\jmath} - 3\hat{k}$

Now,

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

Hence, the two vectors, which are different, having same magnitude.

Question 3:

Write two different vectors having same direction.

Answer 3:

Consider $\vec{p}$ and $\vec{q}$ two different vectors as follows:

$\vec{p} = \hat{\imath} + \hat{\jmath} + 3\hat{k}$ and $\vec{q} = 2\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$

Now the direction cosines of vector $\vec{p}$ are given by

$$l = \frac{1}{\sqrt{1^2 + 1^2 + 3^2}} = \frac{1}{\sqrt{11}}, \quad m = \frac{1}{\sqrt{1^2 + 1^2 + 3^2}} = \frac{1}{\sqrt{11}} \text{ and } n = \frac{3}{\sqrt{1^2 + 1^2 + 3^2}} = \frac{3}{\sqrt{11}}$$

The direction cosines of vector $\vec{q}$ are given by

$$l = \frac{2}{\sqrt{2^2 + 2^2 + 6^2}} = \frac{2}{\sqrt{44}} = \frac{1}{\sqrt{11}}, \; m = \frac{2}{\sqrt{2^2 + 2^2 + 6^2}} = \frac{2}{\sqrt{44}} = \frac{1}{\sqrt{11}}, \; n = \frac{6}{\sqrt{2^2 + 2^2 + 6^2}} = \frac{6}{\sqrt{44}} = \frac{3}{\sqrt{11}}$$

Here, $\vec{p}$ and $\vec{q}$ are different vectors, but they are in same direction as their direction cosines are same.

Question 4:
Find the values of x and y so that the vectors $2\hat{\imath} + 3\hat{\jmath}$ and $x\hat{\imath} + y\hat{\jmath}$ are equal

Answer 4:
The two vectors $2\hat{\imath} + 3\hat{\jmath}$ and $x\hat{\imath} + y\hat{\jmath}$ will be equal if their corresponding components are equal.
Hence, the required values of x and y are 2 and 3 respectively.

Question 5:
Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (–5, 7).

Answer 5:
The vector with the initial point P (2, 1) and terminal point Q (–5, 7) can be given by,
$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (-5\hat{\imath} + 7\hat{\jmath}) - (2\hat{\imath} + \hat{\jmath}) = -7\hat{\imath} + 6\hat{\jmath}$
Hence, the required scalar components are –7 and 6 while the vector components are $-7\hat{\imath}$ and $6\hat{\jmath}$.

Question 6:
Find the sum of the vectors $\vec{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$, $\vec{b} = -2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ and $\vec{c} = \hat{\imath} - 6\hat{\jmath} - 7\hat{k}$.

Answer 6:
The given vectors are $\vec{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$, $\vec{b} = -2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ and $\vec{c} = \hat{\imath} - 6\hat{\jmath} - 7\hat{k}$, therefore
$\vec{a} + \vec{b} + \vec{c}$
$= (\hat{\imath} - 2\hat{\jmath} + \hat{k}) + (-2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}) + (\hat{\imath} - 6\hat{\jmath} - 7\hat{k})$
$= (\hat{\imath} - 2\hat{\imath} + \hat{\imath}) + (-2\hat{\jmath} + 4\hat{\jmath} - 6\hat{\jmath}) + (\hat{k} + 5\hat{k} - 7\hat{k})$
$= 0.\hat{\imath} - 4\hat{\jmath} - \hat{k} = -4\hat{\jmath} - \hat{k}$

Question 7:
Find the unit vector in the direction of the vector $\vec{a} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$.

Answer 7:
The unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

Here, $|\vec{a}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1 + 1 + 4} = \sqrt{6}$, therefore
$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{\imath} + \hat{\jmath} + 2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\hat{\imath} + \frac{1}{\sqrt{6}}\hat{\jmath} + \frac{2}{\sqrt{6}}\hat{k}$$

Question 8:
Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6), respectively.

Answer 8:
The given points are P (1, 2, 3) and Q (4, 5, 6). Therefore,
$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (4\hat{\imath} + 5\hat{\jmath} + 6\hat{k}) - (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) = 3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$
The unit vector in the direction of $\overrightarrow{PQ}$ is given by

$$\widehat{PQ} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$$

Here, $|\overrightarrow{PQ}| = \sqrt{3^2 + 3^2 + 3^2} = \sqrt{9 + 9 + 9} = \sqrt{27} = 3\sqrt{3}$, therefore

$$\widehat{PQ} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}}{3\sqrt{3}} = \frac{\hat{\imath} + \hat{\jmath} + \hat{k}}{\sqrt{3}} = \frac{1}{\sqrt{3}}\hat{\imath} + \frac{1}{\sqrt{3}}\hat{\jmath} + \frac{1}{\sqrt{3}}\hat{k}$$

Question 9:

For given vectors, $\vec{a} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\vec{b} = -\hat{\imath} + \hat{\jmath} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$.

Answer 9:

The given vectors are $\vec{a} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\vec{b} = -\hat{\imath} + \hat{\jmath} - \hat{k}$, therefore

$\vec{a} + \vec{b} = (2\hat{\imath} - \hat{\jmath} + 2\hat{k}) + (-\hat{\imath} + \hat{\jmath} - \hat{k}) = (2\hat{\imath} - \hat{\imath}) + (-\hat{\jmath} + \hat{\jmath}) + (2\hat{k} - \hat{k})$

$= \hat{\imath} + 0.\hat{\jmath} + \hat{k} = \hat{\imath} + \hat{k}$

The unit vector in the direction of $\vec{a} + \vec{b}$ is given by

$$\frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|}$$

Here, $|\vec{a} + \vec{b}| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{1 + 0 + 1} = \sqrt{2}$, therefore the unit vector is given by

$$\frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \frac{\hat{\imath} + \hat{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{k}$$

Question 10:

Find a vector in the direction of vector $5\hat{\imath} - \hat{\jmath} + 2\hat{k}$ which has magnitude 8 units.

Answer 10:

Let $\vec{a} = 5\hat{\imath} - \hat{\jmath} + 2\hat{k}$, therefore $|\vec{a}| = \sqrt{5^2 + (-1)^2 + 2^2} = \sqrt{25 + 1 + 4} = \sqrt{30}$

The unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{5\hat{\imath} - \hat{\jmath} + 2\hat{k}}{\sqrt{30}}$$

Therefore, a vector in the direction of vector $5\hat{\imath} - \hat{\jmath} + 2\hat{k}$ which has magnitude 8 units, is given by

$$8\hat{a} = 8\left(\frac{5\hat{\imath} - \hat{\jmath} + 2\hat{k}}{\sqrt{30}}\right) = \frac{40}{\sqrt{30}}\hat{\imath} - \frac{8}{\sqrt{30}}\hat{\jmath} + \frac{16}{\sqrt{30}}\hat{k}$$

Question 11:

Show that the vectors $2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and $-4\hat{\imath} + 6\hat{\jmath} - 8\hat{k}$ are collinear.

Answer 11:

Let $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and $\vec{b} = -4\hat{\imath} + 6\hat{\jmath} - 8\hat{k}$

Here, $\vec{b} = -4\hat{\imath} + 6\hat{\jmath} - 8\hat{k} = -2(2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}) = -2\vec{a} \Rightarrow \vec{b} = \lambda\vec{a}$, where $\lambda = -2$

Hence, vectors $\vec{a}$ and $\vec{b}$ are collinear.

Question 12:

Find the direction cosines of the vector $\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$.

Answer 12:

Let $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$, therefore

$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

The unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{\imath} + 2\hat{\jmath} + 3\hat{k}}{\sqrt{14}} = \frac{1}{\sqrt{14}}\hat{\imath} + \frac{2}{\sqrt{14}}\hat{\jmath} + \frac{3}{\sqrt{14}}\hat{k}$$

Hence, the direction cosines of the vector $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ are

$$\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$$

Question 13:

Find the direction cosines of the vector joining the points A (1, 2, –3) and B (–1, –2, 1) directed from A to B.

Answer 13:

The given points are A (1, 2, –3) and B (–1, –2, 1). Therefore

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-\hat{\imath} - 2\hat{\jmath} + \hat{k}) - (\hat{\imath} + 2\hat{\jmath} - 3\hat{k}) = -2\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$

The unit vector in the direction of $\overrightarrow{PQ}$ is given by

$$\widehat{AB} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$$

Here, $|\overrightarrow{AB}| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$, therefore

$$\widehat{AB} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{-2\hat{\imath} - 4\hat{\jmath} + 4\hat{k}}{6} = \frac{-\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}{3} = -\frac{1}{3}\hat{\imath} - \frac{2}{3}\hat{\jmath} + \frac{2}{3}\hat{k}$$

Hence, the direction cosines of the vector $\overrightarrow{AB} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ are

$$\left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

Question 14:

Show that the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$ is equally inclined to the axes OX, OY, and OZ.

Answer 14:

Let $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, therefore

$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{1 + 1 + 1} = \sqrt{3}$

The unit vector in the direction of $\vec{a}$ is given by

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{\imath} + \hat{\jmath} + \hat{k}}{\sqrt{3}} = \frac{1}{\sqrt{3}}\hat{\imath} + \frac{1}{\sqrt{3}}\hat{\jmath} + \frac{1}{\sqrt{3}}\hat{k}$$

Hence, the direction cosines of the vector $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ are

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Now, let α, β, and γ be the angles formed by $\vec{a}$ with the positive directions of x, y, and z axes.

Then, we have

$$\cos\alpha = \frac{1}{\sqrt{3}}, \cos\beta = \frac{1}{\sqrt{3}}, \cos\gamma = \frac{1}{\sqrt{3}}$$

Hence, the given vector is equally inclined to axes OX, OY, and OZ.

Question 15:

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $-\hat{\imath} + \hat{\jmath} + \hat{k}$ respectively, in the ration 2:1

(i) internally **(ii)** externally

Answer 15:

The position vector of point R dividing the line segment joining two points P and Q in the ratio m: n is given by:

$$\text{Internally: } \frac{m\vec{b} + n\vec{a}}{m + n}$$

$$\text{Externally: } \frac{m\vec{b} - n\vec{a}}{m - n}$$

Position vectors of P and Q are given as: $\overrightarrow{OP} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $\overrightarrow{OQ} = -\hat{\imath} + \hat{\jmath} + \hat{k}$

(i) The position vector of point R which divides the line joining two points P and Q internally in the ratio 2:1 is given by,

$$\overrightarrow{OR}=\frac{2\left(-\hat{\imath}+\hat{\jmath}+\hat{k}\right)+1\left(\hat{\imath}+2\hat{\jmath}-\hat{k}\right)}{2+1}=\frac{-\hat{\imath}+4\hat{\jmath}+\hat{k}}{3}=-\frac{1}{3}\hat{\imath}+\frac{4}{3}\hat{\jmath}+\frac{1}{3}\hat{k}$$

(ii) The position vector of point R which divides the line joining two points P and Q externally in the ratio 2:1 is given by,

$$\overrightarrow{OR}=\frac{2\left(-\hat{\imath}+\hat{\jmath}+\hat{k}\right)-1\left(\hat{\imath}+2\hat{\jmath}-\hat{k}\right)}{2-1}=\frac{-3\hat{\imath}+0.\hat{\jmath}+3\hat{k}}{1}=-3\hat{\imath}+3\hat{k}$$

Question 16:

Find the position vector of the mid-point of the vector joining the points P (2, 3, 4) and Q (4, 1, – 2).

Answer 16:

The position vector of mid-point R of the vector joining points P (2, 3, 4) and Q (4, 1, – 2) is given by,

$$\overrightarrow{OR}=\frac{\left(2\hat{\imath}+3\hat{\jmath}+4\hat{k}\right)+\left(4\hat{\imath}+\hat{\jmath}-2\hat{k}\right)}{2}$$

$$=\frac{6\hat{\imath}+4\hat{\jmath}+2\hat{k}}{2}$$

$$=3\hat{\imath}+2\hat{\jmath}+\hat{k}$$

Question 17:

Show that the points A, B and C with position vectors, $\vec{a}=3\hat{\imath}-4\hat{\jmath}-4\hat{k}$, $\vec{b}=2\hat{\imath}-\hat{\jmath}+\hat{k}$ and $\vec{c}=\hat{\imath}-3\hat{\jmath}-5\hat{k}$ respectively form the vertices of a right angled triangle.

Answer 17:

Position vectors of points A, B, and C are respectively given as:

$\overrightarrow{OA}=\vec{a}=3\hat{\imath}-4\hat{\jmath}-4\hat{k}$, $\overrightarrow{OB}=\vec{b}=2\hat{\imath}-\hat{\jmath}+\hat{k}$ and $\overrightarrow{OC}=\vec{c}=\hat{\imath}-3\hat{\jmath}-5\hat{k}$

Therefore,

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\vec{b}-\vec{a}=\left(2\hat{\imath}-\hat{\jmath}+\hat{k}\right)-\left(3\hat{\imath}-4\hat{\jmath}-4\hat{k}\right)=-\hat{\imath}+3\hat{\jmath}+5\hat{k}$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\vec{c}-\vec{b}=\left(\hat{\imath}-3\hat{\jmath}-5\hat{k}\right)-\left(2\hat{\imath}-\hat{\jmath}+\hat{k}\right)=-\hat{\imath}-2\hat{\jmath}-6\hat{k}$$

$$\overrightarrow{CA}=\overrightarrow{OA}-\overrightarrow{OC}=\vec{a}-\vec{c}=\left(3\hat{\imath}-4\hat{\jmath}-4\hat{k}\right)-\left(\hat{\imath}-3\hat{\jmath}-5\hat{k}\right)=2\hat{\imath}-\hat{\jmath}+\hat{k}$$

Now,

$$\left|\overrightarrow{AB}\right|=\sqrt{(-1)^2+3^2+5^2}=\sqrt{1+9+25}=\sqrt{35}$$

$$\Rightarrow \left|\overrightarrow{AB}\right|^2=35$$

$$\left|\overrightarrow{BC}\right|=\sqrt{(-1)^2+(-2)^2+(-6)^2}=\sqrt{1+4+36}=\sqrt{41}$$

$$\Rightarrow \left|\overrightarrow{BC}\right|^2=41$$

$$\left|\overrightarrow{CA}\right|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{4+1+1}=\sqrt{6}$$

$$\Rightarrow \left|\overrightarrow{CA}\right|^2=6$$

Here, $\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{CA}\right|^2=35+6=41=\left|\overrightarrow{BC}\right|^2$

Hence, ABC is a right-angled triangle with right angle at A.

Question 18:

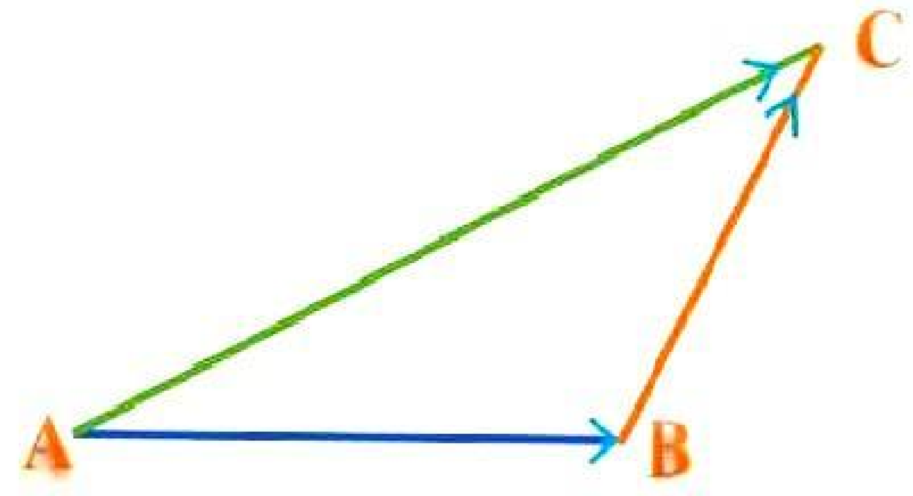
In triangle ABC which of the following is not true:

(A) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$

(B) $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\vec{0}$

(C) $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\vec{0}$

(D) $\overrightarrow{AB}-\overrightarrow{CB}+\overrightarrow{CA}=\vec{0}$



Answer 18:

On applying the triangle law of addition in the given triangle, we have:

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ ... (1)

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}=-\overrightarrow{CA}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$ ... (2)

$\therefore$ The equation given in alternative A is true.

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

$$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\vec{0}$$

$\therefore$ The equation given in alternative B is true,

From equation (2), we have:

$$\overrightarrow{AB}-\overrightarrow{CB}+\overrightarrow{CA}=0$$

$\therefore$The equation given in the alternative D is true.

Now, consider the equation given in alternative C:

$$\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{CA}=\vec{0}$$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{CA}$ ....(3)

From equation (1) and (3), we have:

$$\overrightarrow{AC}=\overrightarrow{CA}$$

$$\Rightarrow \overrightarrow{AC}=-\overrightarrow{AC}$$

$$\Rightarrow \overrightarrow{AC}+\overrightarrow{AC}=0$$

$$\Rightarrow 2\overrightarrow{AC}=\vec{0}$$

$\Rightarrow \overrightarrow{AC}=\vec{0}$ which is not true.

Hence, the equation given in alternative C is **incorrect**.

The correct answer is **C.**

Question 19:

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect:

(A) $\vec{b} = \lambda\vec{a}$, for some scalar λ.

(B) $\vec{a} = \pm\vec{b}$

(C) the respective components of $\vec{a}$ and $\vec{b}$ are proportional.

(D) both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes.

Answer 19:

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then they are parallel.

Therefore, we have: $\vec{b} = \lambda\vec{a}$ (for some scalar λ)
Hence, option (A) is true.

If $\lambda = \pm 1$, we have $\vec{b} = \pm\vec{a}$ or $\vec{a} = \pm\vec{b}$
Hence, the option (B) is also true.

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then they are parallel. The respective components of parallel vectors are proportional.
So, the option (C) is not true.

Hence, the option (D) is correct.

Mathematics

(Chapter – 10) (Vector Algebra) (Exercise 10.3)
(Class – XII)

Question 1:

Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a}.\vec{b} = \sqrt{6}$

Answer 1:

It is given that, $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\vec{a}.\vec{b} = \sqrt{6}$

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

Now, we know that

$\sqrt{6} = \sqrt{3} \times 2 \times \cos\theta$

$\Rightarrow \cos\theta = \frac{\sqrt{6}}{\sqrt{3}\times 2}$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \quad \Rightarrow \theta = \frac{\pi}{4}$

Hence, the angle between the given vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

Question 2:

Find the angle between the vectors $\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ and $3\hat{\imath} - 2\hat{\jmath} + \hat{k}$

Answer 2:

The given vectors are $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ and $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$

$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$

$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$

Now, $\vec{a}.\vec{b} = (\hat{\imath} - 2\hat{\jmath} + 3\hat{k})(3\hat{\imath} - 2\hat{\jmath} + \hat{k})$

$= 1.3 + (-2)(-2) + 3.1 = 3 + 4 + 3 = 10$

Also, we know that $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\therefore 10 = \sqrt{14}\sqrt{14}\cos\theta$

$\Rightarrow \cos\theta = \frac{10}{14} = \frac{5}{7}$ therefore, $\theta = cos^{-1}\left(\frac{5}{7}\right)$

Question 3:

Find the projection of the vector $\hat{\imath} - \hat{\jmath}$ on the vector $\hat{\imath} + \hat{\jmath}$.

Answer 3:

Let $\vec{a} = \hat{\imath} - \hat{\jmath}$ and $\vec{b} = \hat{\imath} + \hat{\jmath}$

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}(\vec{a}.\vec{b}) = \frac{1}{\sqrt{1+1}}[1.1 + (-1)(1)] = \frac{1}{\sqrt{2}}(1-1) = 0$$

Hence, the projection of vector $\vec{a}$ on $\vec{b}$ is 0.

Question 4:

Find the projection of the vector $\hat{\imath} + 3\hat{\jmath} + 7\hat{k}$ on the vector $7\hat{\imath} - \hat{\jmath} + 8\hat{k}$.

Answer 4:

Let $\vec{a} = \hat{\imath} + 3\hat{\jmath} + 7\hat{k}$ and $\vec{b} = 7\hat{\imath} - \hat{\jmath} + 8\hat{k}$

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}(\vec{a}.\vec{b}) = \frac{1}{\sqrt{7^2 + (-1)^2 + 8^2}}[1(7) + 3(-1) + 7(8)] = \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}$$

Question 5:

Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}), \quad \frac{1}{7}(3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}), \quad \frac{1}{7}(6\hat{\imath} + 2\hat{\jmath} - 3\hat{k})$$

Also, show that they are mutually perpendicular to each other.

Answer 5:

Let $\vec{a} = \frac{1}{7}(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}) = \frac{2}{7}\hat{\imath} + \frac{3}{7}\hat{\jmath} + \frac{6}{7}\hat{k}$

$\vec{b} = \frac{1}{7}(3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}) = \frac{3}{7}\hat{\imath} - \frac{6}{7}\hat{\jmath} + \frac{2}{7}\hat{k}$

$\vec{c} = \frac{1}{7}(6\hat{\imath} + 2\hat{\jmath} - 3\hat{k}) = \frac{6}{7}\hat{\imath} + \frac{2}{7}\hat{\jmath} - \frac{3}{7}\hat{k}$

$$|\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = 1$$

$$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(-\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2} = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{4}{49}} = 1$$

$$|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(-\frac{3}{7}\right)^2} = \sqrt{\frac{36}{49} + \frac{4}{49} + \frac{9}{49}} = 1$$

Thus, each of the given three vectors is a unit vector.

$$\vec{a}.\vec{b} = \frac{2}{7} \times \frac{3}{7} + \frac{3}{7} \times \left(\frac{-6}{7}\right) + \frac{6}{7} \times \frac{2}{7} = \frac{6}{49} - \frac{18}{49} + \frac{12}{49} = 0$$

$$\vec{b}.\vec{c} = \frac{3}{7} \times \frac{6}{7} + \left(\frac{-6}{7}\right) \times \frac{2}{7} + \frac{2}{7} \times \left(\frac{-3}{7}\right) = \frac{18}{49} - \frac{12}{49} - \frac{6}{49} = 0$$

$$\vec{c}.\vec{a} = \frac{6}{7} \times \frac{2}{7} + \frac{2}{7} \times \frac{3}{7} + \left(\frac{-3}{7}\right) \times \frac{6}{7} = \frac{12}{49} + \frac{6}{49} - \frac{18}{49} = 0$$

Hence, the given three vectors are mutually perpendicular to each other.

Question 6:

Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.

Answer 6:

$(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 8 \quad \Rightarrow \vec{a}.\vec{a} - \vec{a}.\vec{b} + \vec{b}.\vec{a} - \vec{b}.\vec{b} = 8 \qquad \Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8$

$\Rightarrow (8|\vec{b}|)^2 - |\vec{b}|^2 = 8 \qquad$ [Given that: $|\vec{a}| = 8|\vec{b}|$]

$\Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 = 8$

$\Rightarrow 63|\vec{b}|^2 = 8 \quad$ therefore, $\quad |\vec{b}|^2 = \frac{8}{63}$

$\Rightarrow |\vec{b}| = \sqrt{\frac{8}{63}} \qquad$ [Magnitude of a vector is non- negative]

$\Rightarrow |\vec{b}| = \frac{2\sqrt{2}}{3\sqrt{7}}$

$\Rightarrow |\vec{a}| = 8|\vec{b}| = \frac{8 \times 2\sqrt{2}}{3\sqrt{7}} = \frac{16\sqrt{2}}{3\sqrt{7}}$

Question 7:

Evaluate the product $(3\vec{a} - 5\vec{b}).(2\vec{a} + 7\vec{b})$.

Answer 7:

$(3\vec{a} - 5\vec{b}).(2\vec{a} + 7\vec{b})$

$= 3\vec{a}.2\vec{a} + 3\vec{a}.7\vec{b} - 5\vec{b}.2\vec{a} - 5\vec{b}.7\vec{b}$

$= 6\vec{a}.\vec{a} + 21\vec{a}.\vec{b} - 10\vec{a}.\vec{b} - 35\vec{b}.\vec{b}$

$= 6|\vec{a}|^2 + 11\vec{a}.\vec{b} - 35|\vec{b}|^2$

Question 8:

Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$ having same magnitude and such that the angle between them is 60^0 and their scalar product is $\frac{1}{2}$.

Answer 8:

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$

It is given that $|\vec{a}| = |\vec{b}|, \vec{a}.\vec{b} = \frac{1}{2}$ and $\theta = 60^0$ ...(1)

We know that $\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$\therefore \frac{1}{2} = |\vec{a}||\vec{a}| \cos 60^0$ [using (1)]

$\Rightarrow \frac{1}{2} = |\vec{a}|^2 \times \frac{1}{2}$

$\Rightarrow |\vec{a}|^2 = 1$

$\Rightarrow |\vec{a}| = |\vec{b}| = 1$

Question 9:

Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 12$.

Answer 9:

$(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 12$

$\Rightarrow \vec{x}.\vec{x} + \vec{x}.\vec{a} - \vec{a}.\vec{x} - \vec{a}.\vec{a} = 12$

$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12$ [$|\vec{a}| = 1$ *as* $\vec{a}$ is a unit vector]

$\Rightarrow |\vec{x}|^2 - 1 = 12$

$\Rightarrow |\vec{x}| = \sqrt{13}$

Question 10:

If $\vec{a} = 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$, $\vec{b} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $\vec{c} = 3\hat{\imath} + \hat{\jmath}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

Answer 10:

The given vectors are $\vec{a} = 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$, $\vec{b} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $\vec{c} = 3\hat{\imath} + \hat{\jmath}$

Now, $\vec{a} + \lambda\vec{b} = (2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + \lambda(-\hat{\imath} + 2\hat{\jmath} + \hat{k}) = (2 - \lambda)\hat{\imath} + (2 + 2\lambda)\hat{\jmath} + (3 + \lambda)\hat{k}$

If $(\vec{a} + \lambda\vec{b}).\vec{c} = 0$

$\Rightarrow [(2 - \lambda)\hat{\imath} + (2 + 2\lambda)\hat{\jmath} + (3 + \lambda)\hat{k}].(3\hat{\imath} + \hat{\jmath}) = 0$

$\Rightarrow (2 - \lambda)3 + (2 + 2\lambda)1 + (3 + \lambda)0 = 0$

$\Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0$

$\Rightarrow -\lambda + 8 = 0$

$\Rightarrow \lambda = 8$

Hence, the required value of λ is 8.

Question 11:

Show that: $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$ for any two non zero vectors $\vec{a}$ and $\vec{b}$.

Answer 11:

$(|\vec{a}|\vec{b} + |\vec{b}|\vec{a}).(|\vec{a}|\vec{b} - |\vec{b}|\vec{a})$

$= |\vec{a}|^2\vec{b}.\vec{b} - |\vec{a}||\vec{b}|\vec{b}.\vec{a} + |\vec{b}||\vec{a}|\vec{a}.\vec{b} - |\vec{b}|^2\vec{a}.\vec{a}$

$= |\vec{a}|^2|\vec{b}|^2 - |\vec{b}|^2|\vec{a}|^2 = 0$

Hence, $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ and $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$ are perpendicular to each other.

Question 12:
If $\vec{a}\vec{a} = 0$ and $\vec{a}.\vec{b} = 0$ then what can be concluded about the vector $\vec{b}$?

Answer 12:
It is given that $\vec{a}.\vec{a} = 0$ and $\vec{a}.\vec{b} = 0$

Now, $\vec{a}.\vec{a} = 0$

$\Rightarrow |\vec{a}|^2 = 0$

$\Rightarrow |\vec{a}| = 0$

$\therefore$ $\vec{a}$ is a zero vector.

Hence, vector $\vec{b}$ satisfying $\vec{a}.\vec{b}$ =0 can be any vector.

Question 13:
If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$.

Answer 13:
$$\left|\vec{a} + \vec{b} + \vec{c}\right|^2 = (\vec{a} + \vec{b} + \vec{c}).(\vec{a} + \vec{b} + \vec{c})$$

$$= |\vec{a}|^2 + \left|\vec{b}\right|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})$$

$$\Rightarrow 0 = 1 + 1 + 1 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})$$

$$\Rightarrow (\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = \frac{-3}{2}$$

Question 14:
If either vector $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ then $\vec{a}.\vec{b} = 0$ But the converse need not be true. Justify your answer with an example.

Answer 14:
Consider $\vec{a} = 2\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$ and $\vec{b} = 3\hat{\imath} + 3\hat{\jmath} - 6\hat{k}$

Then , $\vec{a}.\vec{b} = 2.3 + 4.3 + 3(-6) = 6 + 12 - 18 = 0$

We now observe that:

$|\vec{a}| = \sqrt{2^2 + 4^2 + 3^2} = \sqrt{29}$

$\therefore \vec{a} \neq \vec{0}$ so $\left|\vec{b}\right| = \sqrt{3^2 + 3^2 + (-6)^2} = \sqrt{54}$

$\therefore \vec{b} \neq \vec{0}$

Hence, the converse of the given statement need not be true.

Question 15:
If the vertices A, B, C of a triangle ABC are (1, 2, 3), (-1, 0, 0), (0, 1, 2), respectively, then find ∠ABC.
[∠ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$]

Answer 15:
The vertices A, B, C are given as A (1, 2, 3), B (-1, 0, 0), and C (0, 1, 2).

Also, it is given that ∠ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$

$\overrightarrow{BA} = \{1 - (-1)\}\hat{\imath} + (2 - 0)\hat{\jmath} + (3 - 0)\hat{k} = 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$\overrightarrow{BC} = \{0 - (-1)\}\hat{\imath} + (1 - 0)\hat{\jmath} + (2 - 0)\hat{k} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$

$\therefore \overrightarrow{BA}.\overrightarrow{BC} = \left|\overrightarrow{BA}\right|.\left|\overrightarrow{BC}\right| \cos(\angle ABC)$

$\therefore 10 = \sqrt{17} \times \sqrt{6} \cos(\angle ABC)$

$$\Rightarrow \cos(\angle ABC) = \frac{10}{\sqrt{17} \times \sqrt{6}} \quad \Rightarrow \angle ABC = cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

Question 16:
Show that the points A (1, 2, 7), B (2, 6, 3) and C (3, 10, –1) are collinear.

Answer 16:
The given points are A (1, 2, 7), B (2, 6, 3), and C (3, 10, –1).

$\overrightarrow{AB} = (2-1)\hat{\imath} + (6-2)\hat{\jmath} + (3-7)\hat{k} = \hat{\imath} + 4\hat{\jmath} - 4\hat{k}$

$\overrightarrow{BC} = (3-2)\hat{\imath} + (10-6)\hat{\jmath} + (-1-3)\hat{k} = \hat{\imath} + 4\hat{\jmath} - 4\hat{k}$

$\overrightarrow{AC} = (3-1)\hat{\imath} + (10-2)\hat{\jmath} + (-1-7)\hat{k} = 2\hat{\imath} + 8\hat{\jmath} - 8\hat{k}$

$|\overrightarrow{AB}| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$

$|\overrightarrow{BC}| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$

$|\overrightarrow{AC}| = \sqrt{2^2 + 8^2 + (-8)^2} = \sqrt{4+64+64} = \sqrt{132} = 2\sqrt{33}$

$\therefore |\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

Hence, the given points A, B, and C are collinear.

Question 17:
Show that the vectors $2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$ form the vertices of a right angled triangle.

Answer 17:
Let vectors $2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$ be position vectors of points A, B, and C respectively.

i.e. $\overrightarrow{OA} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\overrightarrow{OB} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $\overrightarrow{OC} = 3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$

Now vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ represent the sides of ΔABC

i.e. $\overrightarrow{AB} = (1-2)\hat{\imath} + (-3+1)\hat{\jmath} + (-5-1)\hat{k} = -\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$

$\overrightarrow{BC} = (3-1)\hat{\imath} + (-4+3)\hat{\jmath} + (-4+5)\hat{k} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$

$\overrightarrow{AC} = (2-3)\hat{\imath} + (-1+4)\hat{\jmath} + (1+4)\hat{k} = -\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$

$|\overrightarrow{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$

$|\overrightarrow{BC}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$

$|\overrightarrow{AC}| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{1+9+25} = \sqrt{35}$

$|\overrightarrow{BC}|^2 + |\overrightarrow{AC}|^2 = 6 + 35 = 41 = |\overrightarrow{AB}|^2$

Hence, ΔABC is a right-angled triangle.

Question 18:
If $\vec{a}$ is a nonzero vector of magnitude 'a' and λ a nonzero scalar, then $\lambda\vec{a}$ is unit vector if

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $a = |\lambda|$ (D) $a = \frac{1}{|\lambda|}$

Answer 18:
Vector $\lambda\vec{a}$ is a unit vector if $|\lambda\vec{a}| = 1$

Now, $|\lambda\vec{a}| = 1$

$|\lambda||\vec{a}| = 1$ $[\lambda \neq 0]$

$|\vec{a}| = \frac{1}{|\lambda|}$ $[|\vec{a}| = a]$

Hence, vector $\lambda\vec{a}$ is a unit vector if $a = \frac{1}{|\lambda|}$

The correct answer is D.

Question 1:

Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{\imath} - 7\hat{\jmath} + 7\hat{k}$ and $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$.

Answer 1:

We have

$\vec{a} = \hat{\imath} - 7\hat{\jmath} + 7\hat{k}$ and $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix} = \hat{\imath}(-14 + 14) - \hat{\jmath}(2 - 21) + \hat{k}(-2 + 21) = 19\hat{\jmath} + 19\hat{k}$$

$\therefore |\vec{a} \times \vec{b}| = \sqrt{(19)^2 + (19)^2} = 19\sqrt{2}$

Question 2:

Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ where $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$.

Answer 2:

We have $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

$\therefore \vec{a} + \vec{b} = 4\hat{\imath} + 4\hat{\jmath}$ and $\vec{a} - \vec{b} = 2\hat{\imath} + 4\hat{k}$

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} = \hat{\imath}(16) - \hat{\jmath}(16) + \hat{k}(-8) = 16\hat{\imath} - 16\hat{\jmath} - 8\hat{k}$$

$|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})| = \sqrt{16^2 + (-16)^2 + (-8)^2}$

$= \sqrt{2^2 \times 8^2 + (-2)^2 \times 8^2 + (-8)^2}$

$= 8\sqrt{2^2 + (-2)^2 + 1} = 8\sqrt{9} = 8 \times 3 = 24$

Hence, the unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ given by the relation,

$$= \pm \frac{(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})}{|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})|} = \pm \frac{16\hat{\imath} - 16\hat{\jmath} - 8\hat{k}}{24} = \pm \frac{2\hat{\imath} - 2\hat{\jmath} - \hat{k}}{3} = \pm \frac{2}{3}\hat{\imath} \mp \frac{2}{3}\hat{\jmath} \mp \frac{1}{3}\hat{k}$$

Question 3:

If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{\imath}$, $\frac{\pi}{4}$ with $\hat{\jmath}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\vec{a}$.

Answer 3:

Let unit vector $\vec{a}$ have (a_1, a_2, a_3) components.

Since $\vec{a}$ is a unit vector, $|\vec{a}| = 1$

Also, it is given that $\vec{a}$ makes angle $\frac{\pi}{3}$ with $\hat{\imath}$, $\frac{\pi}{4}$ $\hat{\jmath}$ and an acute angle θ with $\hat{k}$ then,

We have: $\cos\frac{\pi}{3} = \frac{a_1}{|\vec{a}|} \Rightarrow \frac{1}{2} = a_1$ $[|\vec{a}| = 1]$

$\cos\frac{\pi}{4} = \frac{a_2}{|\vec{a}|} \Rightarrow \frac{1}{\sqrt{2}} = a_2$ $[|\vec{a}| = 1]$

And also $\cos\theta = \frac{a_3}{|\vec{a}|} \Rightarrow a_3 = \cos\theta$

Now, $|a| = 1$

$\Rightarrow \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2} = 1 \Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + cos^2\theta = 1$

$\Rightarrow \frac{1}{4} + \frac{1}{2} + cos^2\theta = 1$ $\Rightarrow \frac{3}{4} + cos^2\theta = 1$

$\Rightarrow cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$ $\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$

$\therefore a_3 = \cos\frac{\pi}{3} = \frac{1}{2}$

Hence $\theta = \frac{\pi}{3}$ and the components of $\vec{a}$ are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

Question 4:

$(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b})$

Answer 4:

$$LHS=\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)$$
$$=\left(\vec{a}-\vec{b}\right)\times\vec{a}+\left(\vec{a}-\vec{b}\right)\times\vec{b}$$
$$=\vec{a}\times\vec{a}-\vec{b}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{b}$$
$$=0+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-0$$
$$=2\vec{a}\times\vec{b}$$
$$=RHS$$

Question 5:

Find λ and μ if $\left(2\hat{\imath}+6\hat{\jmath}+27\hat{k}\right)\times\left(\hat{\imath}+\lambda\hat{\jmath}+\mu\hat{k}\right)=\vec{0}$.

Answer 5:

: $\left(2\hat{\imath}+6\hat{\jmath}+27\hat{k}\right)\times\left(\hat{\imath}+\lambda\hat{\jmath}+\mu\hat{k}\right)=\vec{0}$

$$\Rightarrow\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 6 & 27\\ 1 & \lambda & \mu\end{vmatrix}=0\hat{\imath}+0\hat{\jmath}+0\hat{k}$$

$\Rightarrow\ \hat{\imath}\,(6\mu-27\lambda)-\hat{\jmath}\ (2\mu-27)+\hat{k}\ (2\lambda-6)=0\hat{\imath}+0\hat{\jmath}+0\hat{k}$

On comparing the corresponding components, we have: $6\mu-27\lambda=0 \Rightarrow 2\mu-27=0$

$\Rightarrow 2\lambda-6=0$

Now, $2\lambda-6=0\Rightarrow\lambda=3$

$\Rightarrow 2\mu-27=0\ \Rightarrow\mu=\dfrac{27}{2}$

Hence , $\lambda=3$ and $\mu=\frac{27}{2}$

Question 6:

Given that $\vec{a}.\vec{b}=0$ and $\vec{a}\times\vec{b}=\vec{0}$

What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?

Answer 6:

If $\vec{a}.\vec{b}=0$, then, either $|\vec{a}|=0$ or $|\vec{b}|=0$ or $\vec{a}\perp\vec{b}$ (In case $\vec{a}$ and $\vec{b}$ are non zero)

If $\vec{a}\times\vec{b}=0$ then, either $|\vec{a}|=0$ or $|\vec{b}|=0$ or $\vec{a}\parallel\vec{b}$ (In case $\vec{a}$ and $\vec{b}$ are non zero)

But $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

Hence, $|\vec{a}|=0$ or $|\vec{b}|=0$

Question 7:

Let the vectors $\vec{a},\vec{b},\vec{c}$ be given as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$. Then show that $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.

Answer 7:

$\vec{a}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$, $\vec{b}=b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k}$ $\vec{c}=c_1\hat{\imath}+c_2\hat{\jmath}+c_3\hat{k}$

$(\vec{b}+\vec{c})=(b_1+c_1)\hat{\imath}+(b_2+c_2)\hat{\jmath}+(b_3+c_3)\hat{k}$

$$\text{Now, }\vec{a}\times(\vec{b}+\vec{c})=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}$$

$=\hat{\imath}\,[a_2(b_3+c_3)-a_3(b_2+c_2)]-\hat{\jmath}\,[a_1(b_3+c_3)-a_3(b_1+c_1)]+\hat{k}\,[a_1(b_2+c_2)-a_2(b_1+c_1)]$

$=\hat{\imath}\,[a_2b_3+a_2c_3-a_3b_2-a_3c_2]-\hat{\jmath}\,[a_1b_3+a_1c_3-a_3b_1-a_3c_1]+\hat{k}\,[a_1b_2+a_1c_2-a_2b_1-a_2c_1]$ ...(1)

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\hat{\imath}\,[a_2b_3-a_3b_2]+\hat{\jmath}\,[a_3b_1-a_1b_3]+\hat{k}\,[a_1b_2-a_2b_1] \quad \ldots (2)$$

$$\vec{a}\times\vec{c}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}=\hat{\imath}\,[a_2c_3-a_3c_2]+\hat{\jmath}\,[a_3c_1-a_1c_3]+\hat{k}\,[a_1c_2-a_2c_1] \quad \ldots (3)$$

On adding (2) and (3), we get: $(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$

$$=\hat{\imath}\,[a_2b_3+a_2c_3-a_3c_2-a_3b_2]+\hat{\jmath}\,[a_3b_1+a_3c_1-a_1b_3-a_1c_3]$$
$$+\hat{k}\,[a_1b_2+a_1c_2-a_2b_1-a_2c_1] \quad \ldots (4)$$

Now from (1) and (4), we have

$$\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

Hence, the given result is proved.

Question 8:

If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$ Then $\vec{a}\times\vec{b}=\vec{0}$ Is the converse true? Justify your answer with an example.

Answer 8:

Take any parallel non-zero vectors so that $\vec{a}\times\vec{b}=\vec{0}$

Let $\vec{a}=2\hat{\imath}+3\hat{\jmath}+4\hat{k}$, $\vec{b}=4\hat{\imath}+6\hat{\jmath}+8\hat{k}$

Then, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 3 & 4\\ 4 & 6 & 8\end{vmatrix}=\hat{\imath}(24-24)-\hat{\jmath}(16-16)+\hat{k}(12-12)=0\hat{\imath}-0\hat{\jmath}-0\hat{k}=\vec{0}$

It can now be observed that:

$|\vec{a}|=\sqrt{2^2+3^2+4^2}=\sqrt{29}$

$\therefore \vec{a}\neq\vec{0}$

$|\vec{b}|=\sqrt{4^2+6^2+8^2}=\sqrt{116}$

$\therefore \vec{b}\neq\vec{0}$

Hence, the converse of the given statement need not be true.

Question 9:

Find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

Answer 9:

The vertices of triangle ABC are given as A (1, 1, 2), B (2, 3, 5), and C (1, 5, 5).

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of ΔABC are given as:

$\overrightarrow{AB}=(2-1)\hat{\imath}+(3-1)\hat{\jmath}+(5-2)\hat{k}=\hat{\imath}+2\hat{\jmath}+3\hat{k}$

$\overrightarrow{BC}=(1-2)\hat{\imath}+(5-3)\hat{\jmath}+(5-5)\hat{k}=-\hat{\imath}+2\hat{\jmath}$

Area of ΔABC $=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|$

$\overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 2 & 3\\ -1 & 2 & 0\end{vmatrix}=\hat{\imath}(-6)-\hat{\jmath}(3)+\hat{k}(2+2)=-6\hat{\imath}-3\hat{\jmath}+4\hat{k}$

$\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=\sqrt{(-6)^2+(-3)^2+4^2}=\sqrt{36+9+16}=\sqrt{61}$

Hence, the area of ΔABC is $\frac{\sqrt{61}}{2}$ square units.

Question 10:

Find the area of the parallelogram whose adjacent sides are determined by the vector $\vec{a}=\hat{\imath}-\hat{\jmath}+3\hat{k}$ and $\vec{b}=2\hat{\imath}-7\hat{\jmath}+\hat{k}$

Answer 10:

The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$

Adjacent sides are given as: $\vec{a}=\hat{\imath}-\hat{\jmath}+3\hat{k}$ and $\vec{b}=2\hat{\imath}-7\hat{\jmath}+\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{vmatrix}=\hat{\imath}(-1+21)-\hat{\jmath}(1-6)+\hat{k}(-7+2)=20\hat{\imath}+5\hat{\jmath}-5\,\hat{k}$$

$$|\vec{a}\times\vec{b}|=\sqrt{(20)^2+(5)^2+(-5)^2}=\sqrt{400+25+25}=15\sqrt{2}$$

Hence, the area of the given parallelogram is $15\sqrt{2}$ square units.

Question 11:

Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$ then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Answer 11:

It is given that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$

We know that $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$.

Now, $\vec{a}\times\vec{b}$ is a unit vector if $|\vec{a}\times\vec{b}|=1$, so

$$|\vec{a}\times\vec{b}|=1\Rightarrow\left||\vec{a}||\vec{b}|\sin\theta\,\hat{n}\right|=1\qquad\Rightarrow\left||\vec{a}||\vec{b}|\sin\theta\right|=1$$

$$\Rightarrow 3\times\frac{\sqrt{2}}{3}\times\sin\theta=1\ \Rightarrow\sin\theta=\frac{1}{\sqrt{2}}\quad\Rightarrow\theta=\frac{\pi}{4}$$

Hence $\vec{a}\times\vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$

The correct answer is (B).

Question 12:

Area of a rectangle having vertices A, B, C, and D with position vectors $-\hat{\imath}+\frac{1}{2}\hat{\jmath}+4\hat{k}$, $\hat{\imath}+\frac{1}{2}\hat{\jmath}+4\hat{k}$, $\hat{\imath}-\frac{1}{2}\hat{\jmath}+4\hat{k}$ and $-\hat{\imath}-\frac{1}{2}\hat{\jmath}+4\hat{k}$ respectively is:

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

Answer 12:

The position vectors of vertices A, B, C, and D of rectangle ABCD are given as:

$-\hat{\imath}+\frac{1}{2}\hat{\jmath}+4\hat{k}$, $\hat{\imath}+\frac{1}{2}\hat{\jmath}+4\hat{k}$, $\hat{\imath}-\frac{1}{2}\hat{\jmath}+4\hat{k}$ and $-\hat{\imath}-\frac{1}{2}\hat{\jmath}+4\hat{k}$ respectively

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of the given rectangle are given as:

$$\overrightarrow{AB}=(1+1)\hat{\imath}+\left(\frac{1}{2}-\frac{1}{2}\right)\hat{\jmath}+(4-4)\hat{k}=2\hat{\imath}$$

$$\overrightarrow{BC}=(1-1)\hat{\imath}+\left(-\frac{1}{2}-\frac{1}{2}\right)\hat{\jmath}+(4-4)\hat{k}=-\hat{\jmath}$$

$$\therefore\overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 0 & 0\\ 0 & -1 & 0\end{vmatrix}=-2\hat{k}$$

$$|\overrightarrow{AB}\times\overrightarrow{BC}|=\sqrt{(-2)^2}=2$$

Now, it is known that the area of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$.

Hence, the area of the given rectangle is $|\overrightarrow{AB}\times\overrightarrow{BC}|=2$ square units.

The correct answer is (C).

Mathematics

(Chapter – 10) (Vector Algebra) (Miscellaneous Exercise)
(Class – XII)

Question 1:
Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.

Answer 1:
If $\vec{r}$ is a unit vector in the XY- plane , then $\vec{r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

Here, θ is the angle made by the unit vector with the positive direction of the x-axis.

Therefore, for θ = 30°:

$$\vec{r} = \cos 30^0\,\hat{\imath} + \sin 30^0\,\hat{\jmath} = \frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$$

Hence, the required unit vector is $\frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$

Question 2:
Find the scalar components and magnitude of the vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

Answer 2:
The vector joining the point $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ $\overrightarrow{PQ}$ = Position vector of Q − Position vector of P

$= (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$

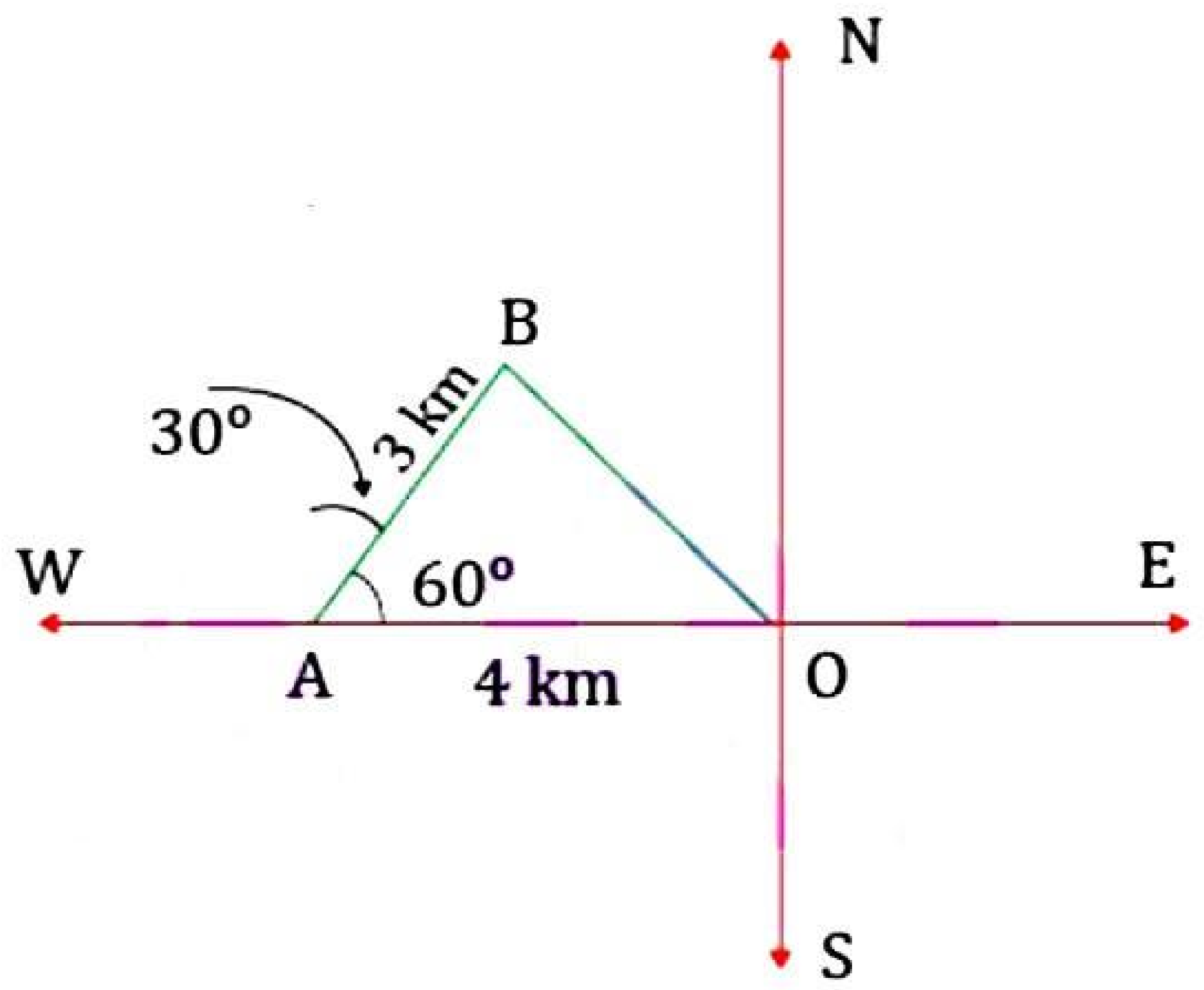
$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Hence, the scalar components and the magnitude of the vector joining the given points are respectively $\{(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)\}$ and $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Question 3:
A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Answer 3:
Let O and B be the initial and final positions of the girl respectively.

Then, the girl's position can be shown as:



Now we have , $\overrightarrow{OA} = -4\hat{\imath}$

$$\overrightarrow{AB} = \hat{\imath}|\overrightarrow{AB}|\cos 60^0 + \hat{\jmath}|\overrightarrow{AB}|\sin 60^0$$

$$= \hat{\imath}3 \times \frac{1}{2} + \hat{\jmath}3 \times \frac{\sqrt{3}}{2}$$

$$= \frac{3}{2}\hat{\imath} + \frac{3\sqrt{3}}{2}\hat{\jmath}$$

By the triangle law of vector addition, we have:

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$$\Rightarrow (-4\hat{\imath}) + \left(\frac{3}{2}\hat{\imath} + \frac{3\sqrt{3}}{2}\hat{\jmath}\right)$$

$$= \left(-4 + \frac{3}{2}\right)\hat{\imath} + \left(\frac{3\sqrt{3}}{2}\right)\hat{\jmath}$$

$$\Rightarrow \left(\frac{-8+3}{2}\right)\hat{\imath} + \left(\frac{3\sqrt{3}}{2}\right)\hat{\jmath}$$

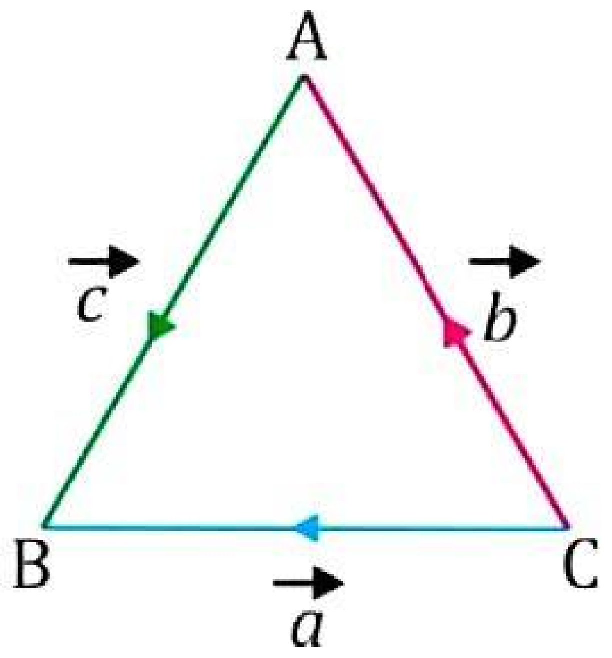
$$= \frac{-5}{2}\hat{\imath} + \frac{3\sqrt{3}}{2}\hat{\jmath}$$

Hence, the girl's displacement from her initial point of departure is $\frac{-5}{2}\hat{\imath} + \frac{3\sqrt{3}}{2}\hat{\jmath}$

Question 4:

If $\vec{a} = \vec{b} + \vec{c}$ then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.

Answer 4:



In ΔABC, Let $\overrightarrow{CB} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, and $\overrightarrow{AB} = \vec{c}$
(as shown in the following figure)
Now, by the triangle law of vector addition, we have $\vec{a} = \vec{b} + \vec{c}$.
It is clearly known that $|\vec{a}|$, $|\vec{b}|$ *and* $|\vec{c}|$ represent the sides of ΔABC.
Also, it is known that the sum of the lengths of any two sides of a triangle is greater than the third side.

$$|\vec{a}| < |\vec{b}| + |\vec{c}|$$

Hence, it is not true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$

Question 5:

Find the value of x for which $x(\hat{\imath} + \hat{\jmath} + \hat{k})$ is a unit vector .

Answer 5:

$x(\hat{\imath} + \hat{\jmath} + \hat{k})$ is a unit vector if $|x(\hat{\imath} + \hat{\jmath} + \hat{k})| = 1$
Now, $|x(\hat{\imath} + \hat{\jmath} + \hat{k})| = 1$

$\Rightarrow \sqrt{x^2 + x^2 + x^2} = 1$

$\Rightarrow \sqrt{3x^2} = 1 \;\Rightarrow\; x = \pm\dfrac{1}{\sqrt{3}}$

Hence, the required value of x is $\pm\frac{1}{\sqrt{3}}$.

Question 6:

Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$ and $\vec{b} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$.

Answer 6:

We have, $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$ and $\vec{b} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$

Let $\vec{c}$ be the resultant of $\vec{a}$ and $\vec{b}$

Then, $\vec{c} = \vec{a} + \vec{b} = (2+1)\hat{\imath} + (3-2)\hat{\jmath} - (-1+1)\hat{k} = 3\hat{\imath} + \hat{\jmath}$

$\therefore |\vec{c}| = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$

$\therefore \hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{(3\hat{\imath}+\hat{\jmath})}{\sqrt{10}}$

Hence, the vector of magnitude 5 units and parallel to the resultant of vectors $\vec{a}$ *and* $\vec{b}$ is

$\pm 5.\hat{c} = \pm 5.\dfrac{1}{\sqrt{10}}(3\hat{\imath} + \hat{\jmath}) = \pm\dfrac{3\sqrt{10}\,\hat{\imath}}{2} \pm \dfrac{\sqrt{10}\,\hat{\jmath}}{2}$

Question 7:

If $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $\vec{c} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

Answer 7:

we have, $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $\vec{c} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$

$2\vec{a} - \vec{b} + 3\vec{c} = 2(\hat{\imath} + \hat{\jmath} + \hat{k}) - (2\hat{\imath} - \hat{\jmath} + 3\hat{k}) + 3(\hat{\imath} - 2\hat{\jmath} + \hat{k})$

$= 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k} - 2\hat{\imath} + \hat{\jmath} - 3\hat{k} + 3\hat{\imath} - 6\hat{\jmath} + 3\hat{k} = 3\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$

$[2\vec{a} - \vec{b} + 3\vec{c}] = \sqrt{3^2 + (-3)^2 + 2^2} = \sqrt{9+9+4} = \sqrt{22}$

Hence, the unit vector along $2\vec{a} - \vec{b} + 3\vec{c}$ is given by

$= \dfrac{2\vec{a} - \vec{b} + 3\vec{c}}{[2\vec{a} - \vec{b} + 3\vec{c}]} = \dfrac{3\hat{\imath} - 3\hat{\jmath} + 2\hat{k}}{\sqrt{22}} = \dfrac{3}{\sqrt{22}}\hat{\imath} - \dfrac{3}{\sqrt{22}}\hat{\jmath} + \dfrac{2}{\sqrt{22}}\hat{k}$

Question 8:

Show that the points A (1, –2, –8), B (5, 0, –2) and C (11, 3, 7) are collinear, and find the ratio in which B divides AC.

Answer 8:

The given points are A (1, –2, –8), B (5, 0, –2), and C (11, 3, 7).

$\overrightarrow{AB} = (5-1)\hat{\imath} + (0+2)\hat{\jmath} + (-2+8)\hat{k} = 4\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$

$\overrightarrow{BC} = (11-5)\hat{\imath} + (3-0)\hat{\jmath} + (7+2)\hat{k} = 6\hat{\imath} + 3\hat{\jmath} + 9\hat{k}$

$\overrightarrow{AC} = (11-1)\hat{\imath} + (3+2)\hat{\jmath} + (7+8)\hat{k} = 10\hat{\imath} + 5\hat{\jmath} + 15\hat{k}$

$|\overrightarrow{AB}| = \sqrt{4^2+2^2+6^2} = \sqrt{16+4+36} = \sqrt{56} = 2\sqrt{14}$

$|\overrightarrow{BC}| = \sqrt{6^2+3^2+9^2} = \sqrt{36+9+81} = \sqrt{126} = 3\sqrt{14}$

$|\overrightarrow{AC}| = \sqrt{(10)^2+5^2+(15)^2} = \sqrt{100+25+225} = \sqrt{350} = 5\sqrt{14}$

$\therefore |\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

Thus, the given points A, B, and C are collinear.

Now, let point B divide AC in the ratio $\lambda : 1$ then, we have

$|\overrightarrow{OB}| = \frac{\lambda\overrightarrow{OC}}{(\lambda+1)}$

$\Rightarrow 5\hat{\imath} - 2\hat{k} = \dfrac{\lambda(11\hat{\imath} + 3\hat{\jmath} + 7\hat{k}) + (\hat{\imath} - 2\hat{\jmath} - 8\hat{k})}{(\lambda+1)}$

$\Rightarrow (\lambda+1)(5\hat{\imath} - 2\hat{k}) = 11\lambda\,\hat{\imath} + 3\lambda\,\hat{\jmath} + 7\lambda\,\hat{k} + \hat{\imath} - 2\hat{\jmath} - 8\hat{k}$

$\Rightarrow 5(\lambda+1)\hat{\imath} - 2(\lambda+1)\hat{k} = (11\lambda+1)\,\hat{\imath} + (3\lambda-2)\,\hat{\jmath} + (7\lambda-8)\,\hat{k}$

On equating the corresponding components, we get:

$5(\lambda+1) = 11\lambda + 1$

$5\lambda + 5 = 11\lambda + 1 \quad \Rightarrow 6\lambda = 4 \Rightarrow \lambda = \frac{4}{6} = \frac{2}{3}$

Hence, point B divides AC in the ratio 2:3

Question 9:

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ externally in the ratio 1: 2. Also, show that P is the midpoint of the line segment RQ.

Answer 9:

It is given that $\overrightarrow{OP} = 2\vec{a} + \vec{b},\ \overrightarrow{OQ} = \vec{a} - 3\vec{b}$

It is given that point R divides a line segment joining two points P and Q externally in the ratio 1: 2. Then, on using the section formula, we get:

$\overrightarrow{OR} = \frac{2(2\vec{a}+\vec{b})-(\vec{a}-3\vec{b})}{2-1} = \frac{4\vec{a}+2\vec{b}-\vec{a}+3\vec{b}}{1} = 3\vec{a} + 5\vec{b}$

Therefore, the position vector of point R is $3\vec{a} + 5\vec{b}$

Position vector of the mid-point of RQ $= \frac{\overrightarrow{OQ}+\overrightarrow{OR}}{2}$

$= \frac{(\vec{a}-3\vec{b})+(3\vec{a}+5\vec{b})}{2} = 2\vec{a} - \vec{b} = \overrightarrow{OP}$

Hence, P is the mid-point of the line segment RQ.

Question 10:

The two adjacent sides of a parallelogram are $2\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$ and $\hat{\imath} - 2\hat{\jmath} - 3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

Answer 10:

Adjacent sides of a parallelogram are given as: $\vec{a} = 2\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$ and $\vec{b} = \hat{\imath} - 2\hat{\jmath} - 3\hat{k}$

Then, the diagonal of a parallelogram is given by $\vec{a} + \vec{b} = (2+1)\hat{\imath} + (-4-2)\hat{\jmath} + (5-3)\hat{k} = 3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}$

Thus, the unit vector parallel to the diagonal is

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}=\frac{3\hat{\imath}-6\hat{\jmath}+2\hat{k}}{\sqrt{3^2+(-6)^2+2^2}}=\frac{3\hat{\imath}-6\hat{\jmath}+2\hat{k}}{\sqrt{9+36+4}}=\frac{3\hat{\imath}-6\hat{\jmath}+2\hat{k}}{7}=\frac{3}{7}\,\hat{\imath}-\frac{6}{7}\,\hat{\jmath}+\frac{2}{7}\,\hat{k}$$

$\therefore$ Area of parallelogram ABCD $=|\vec{a}\times\vec{b}|$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & -4 & 5\\ 1 & -2 & -3\end{vmatrix}=\hat{\imath}(12+10)-\hat{\jmath}(-6-5)+\hat{k}(-4+4)$$

$=22\hat{\imath}+11\hat{\jmath}\quad=11(2\hat{\imath}+\hat{\jmath})$

$\therefore|\vec{a}\times\vec{b}|=11\sqrt{2^2+1^2}=11\sqrt{5}$

Hence, the area of the parallelogram is $11\sqrt{5}$ square units.

Question 11:

Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\pm\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

Answer 11:

Let a vector be equally inclined to OX, OY and OZ at an angle α

So, the DCs of the vectors are $\cos\alpha$, $\cos\alpha$ and $\cos\alpha$.

Therefore,

$$\cos^2\alpha+\cos^2\alpha+\cos^2\alpha=1$$
$$\Rightarrow 3\cos^2\alpha=1$$
$$\Rightarrow \cos^2\alpha=\frac{1}{3}$$
$$\Rightarrow \cos\alpha=\pm\frac{1}{\sqrt{3}}$$

Thus, the DCs of the vector are $\pm\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.

Question 12:

Let $\vec{a}=\hat{\imath}+4\hat{\jmath}+2\hat{k},\ \vec{b}=3\hat{\imath}-2\hat{\jmath}+7\hat{k}$, and $\vec{c}=2\hat{\imath}-\hat{\jmath}+4\hat{k}$ Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}.\vec{d}=15$.

Answer 12:

Let $\vec{d}=d_1\hat{\imath}+d_2\hat{\jmath}+d_3\hat{k}$

Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

$\vec{d}.\vec{a}=0\quad\Rightarrow d_1+4d_2+2d_3=0\qquad\ldots(i)$

And, $\vec{d}.\vec{b}=0\quad\Rightarrow 3d_1-2d_2+7d_3=0\qquad\ldots(ii)$

Also, it is given that: $\vec{c}.\vec{d}=15$

$\Rightarrow 2d_1-d_2+4d_3=15\qquad\ldots(iii)$

On solving (i), (ii), and (iii), we get:

$d_1=\frac{160}{3},\ d_2=-\frac{5}{3},$ and $d_3=-\frac{70}{3}$

$\therefore\vec{d}=\frac{160}{3}\hat{\imath}-\frac{5}{3}4\hat{\jmath}-\frac{70}{3}\hat{k}=\frac{1}{3}\left(160\hat{\imath}-5\hat{\jmath}-70\hat{k}\right)$

Hence, the required vector is $\frac{1}{3}\left(160\hat{\imath}-5\hat{\jmath}-70\hat{k}\right)$.

Question 13:

The scalar product of the vector $\hat{\imath}+\hat{\jmath}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{\imath}+4\hat{\jmath}-5\hat{k}$ *and* $\lambda\,\hat{\imath}+2\hat{\jmath}+3\hat{k}$ is equal to one. Find the value of λ.

Answer 13:

$\left(2\hat{\imath}+4\hat{\jmath}-5\hat{k}\right)+\left(\lambda\,\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)=(2+\lambda)\hat{\imath}+6\hat{\jmath}-2\hat{k}$

Therefore, unit vector along $\left(2\hat{\imath}+4\hat{\jmath}-5\hat{k}\right)+\left(\lambda\,\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)$ is given as:

$$=\frac{(2+\lambda)\hat{\imath}+6\hat{\jmath}-2\hat{k}}{\sqrt{(2+\lambda)^2+(6)^2+(-2)^2}}=\frac{(2+\lambda)\hat{\imath}+6\hat{\jmath}-2\hat{k}}{\sqrt{4+4\lambda+\lambda^2+36+4}}=\frac{(2+\lambda)\hat{\imath}+6\hat{\jmath}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}$$

Scalar product of $(\hat{\imath}+\hat{\jmath}+\hat{k})$ with this unit vector is 1.

$\Rightarrow (\hat{\imath}+\hat{\jmath}+\hat{k}).\dfrac{(2+\lambda)\hat{\imath}+6\hat{\jmath}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}=1$

$\Rightarrow \dfrac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}}=1 \quad \Rightarrow \sqrt{\lambda^2+4\lambda+44}=\lambda+6$

$\Rightarrow \lambda^2+4\lambda+44=(\lambda+6)^2$

$\Rightarrow \lambda^2+4\lambda+44=\lambda^2+12\lambda+36$

$\Rightarrow 8\lambda=8 \quad \Rightarrow \lambda=1$

Hence, the value of λ is 1.

Question 14:

If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a},\vec{b}$ and $\vec{c}$

Answer 14:

Since $\vec{a},\vec{b}$ *and* $\vec{c}$ are mutually perpendicular vectors, we have

$\vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}=0.$

It is given that $|\vec{a}|=|\vec{b}|=|\vec{c}|$

Let vector $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$ at angles , $\theta_1,\theta_2,$ and θ_3 respectively.

Then we have,

$$=\cos\theta_1=\frac{(\vec{a}+\vec{b}+\vec{c}).\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}=\frac{\vec{a}.\vec{a}+\vec{b}.\vec{a}+\vec{c}.\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$=\frac{|\vec{a}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} \qquad [\vec{b}.\vec{a}=\vec{c}.\vec{a}=0]$$

$$=\frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$=\cos\theta_2=\frac{(\vec{a}+\vec{b}+\vec{c}).\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|}=\frac{\vec{a}.\vec{b}+\vec{b}.\vec{b}+\vec{c}.\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|}$$

$$=\frac{|\vec{b}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|} \qquad [\vec{a}.\vec{b}=\vec{c}.\vec{b}=0]$$

$$=\frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$=\cos\theta_3=\frac{(\vec{a}+\vec{b}+\vec{c}).\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}=\frac{\vec{a}.\vec{c}+\vec{b}.\vec{c}+\vec{c}.\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}$$

$$=\frac{|\vec{c}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|} \qquad [\vec{a}.\vec{c}=\vec{b}.\vec{c}=0]$$

$$=\frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

Now, as $|\vec{a}|=|\vec{b}|=|\vec{c}|$ $\cos\theta_1=\cos\theta_2=\cos\theta_3$

$\therefore \theta_1=\theta_2=\theta_3$

Hence, the vector $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Question 15:

Prove that, $(\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$ if and only if $\vec{a}$, $\vec{b}$ are perpendicular, given $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$.

Answer 15:

$(\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$

$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = |\vec{a}|^2 + |\vec{b}|^2$ [Distributive of scalar product over addition]

$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$ [Scalar product is commutative]

$\Rightarrow 2\vec{a}.\vec{b} = 0 \quad \Rightarrow \vec{a}.\vec{b} = 0$

$\therefore$ $\vec{a}$ and $\vec{b}$ are perpendicular. $\quad \vec{a} \neq \vec{0},\ \vec{b} \neq \vec{0}$ (Given)

Choose the correct answer in Exercises 16 to 19.

Question 16:

If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}.\vec{b} \geq 0$ only when

(A) $0 < \theta < \frac{\pi}{2}$ (B) $0 \leq \theta \leq \frac{\pi}{2}$ (C) $0 < \theta < \pi$ (D) $0 \leq \theta \leq \pi$

Answer 16:

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality $\vec{a}$ and $\vec{b}$ are non- zero vectors so that $|\vec{a}|$ and $|\vec{b}|$ are positive.

It is known that $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\therefore \vec{a}.\vec{b} \geq 0 \Rightarrow |\vec{a}||\vec{b}|\cos\theta \geq 0$

$\Rightarrow \cos\theta \geq 0$ [Because $|\vec{a}|$ and $|\vec{b}|$ are positive]

$\Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$

Hence $\vec{a}.\vec{b} \geq 0$ when $0 \leq \theta \leq \frac{\pi}{2}$

The correct answer is (B).

Question 17:

Let $\vec{a}$ *and* $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a} + \vec{b}$ is a unit vector if

(A) $\theta = \frac{\pi}{4}$ (B) $\theta = \frac{\pi}{3}$ (C) $\theta = \frac{\pi}{2}$ (D) $\theta = \frac{2\pi}{3}$

Answer 17:

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ be the angle between them

Then, $|\vec{a}| = |\vec{b}| = 1$

Now, $\vec{a} + \vec{b}$ is a unit vector if $|\vec{a} + \vec{b}| = 1$

$|\vec{a} + \vec{b}| = 1$

$\Rightarrow (\vec{a} + \vec{b})^2 = 1$

$\Rightarrow (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = 1$

$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = 1$

$\Rightarrow |\vec{a}|^2 + 2\vec{a}\vec{b} + |\vec{b}|^2 = 1$

$\Rightarrow 1^2 + 2|\vec{a}||\vec{b}|\cos\theta + 1^2 = 1$

$\Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \frac{2\pi}{3}$

Hence $\vec{a} + \vec{b}$ is a unit vector if $\theta = \frac{2\pi}{3}$

The correct answer is (D).

Question 18:

The value of $\hat{\imath}.(\hat{\jmath} \times \hat{k}) + \hat{\jmath}.(\hat{\imath} \times \hat{k}) + \hat{k}.(\hat{\imath} \times \hat{\jmath})$ is

(A) 0 (B) -1 (C) 1 (D) 3

Answer 18:

: $\hat{\imath}(\hat{\jmath} \times \hat{k}) + \hat{\jmath}(\hat{\imath} \times \hat{k}) + \hat{k}(\hat{\imath} \times \hat{\jmath})$

$= \hat{\imath}.\hat{\imath} + \hat{\jmath}(-\hat{\jmath}) + \hat{k}.\hat{k}$

$\Rightarrow 1 - \hat{\jmath}.\hat{\jmath} + 1$

$\Rightarrow 1 - 1 + 1 = 1$

The correct answer is (C).

Question 19:

If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$ then $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π

Answer 19:

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$

Then, without loss of generality $\vec{a}$ and $\vec{b}$ are non-zero vectors, so that $|\vec{a}|$ and $|\vec{b}|$ are positive $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$

$\Rightarrow |\vec{a}||\vec{b}| \cos\theta = |\vec{a}||\vec{b}| \sin\theta$

$\Rightarrow \cos\theta = \sin\theta$ [$|\vec{a}|$ and $|\vec{b}|$ are positive]

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \dfrac{\pi}{4}$

Hence, $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$, when θ is equal to $\frac{\pi}{4}$.

The correct answer is (B).